UNIT 7 INEQUALITIES

Structure

7.0 INTRODUCTION

In this unit, you will learn about some inequalities and their basic properties. The Arithmetic Mean and Geometric Mean are one of the fundamental inequalities in Algebra, and are used extensively in Olympiad mathematics to solve many problems. The aim of this unit is to acquaint students with the inequality, its proof and various applications. The arithmetic mean of n numbers, better known as the average of n numbers is an example of a mathematical concept that comes up in everyday conversation. A less commonly known mean is the geometric mean. In this unit, you will investigate the geometric mean, derive the arithmetic mean-geometric mean (AM-GM) inequality and do challenging problems.

7.1 OBJECTIVES

After going through this unit, you will be able to:

- Define inequalities
- Discuss properties of inequalities
- Describe Weierstress inequality

7.2 INEQUATIES AND BASIC PROPERTIES

Let a and b be two real numbers. Then $a > b$ if $a–b$ is a positive real number. $a \geq b$ if $a > b$ or $a = b$. $a \leq b$ if $a < b$ or $a = b$.

Basic Properties

1. If $a > b$ and x is any real number, then $a + x > b + x$ and $a - x > b - x$.
2. If $a_1 > b_1, a_2 > b_2, \ldots a_n > b_n$ then $a_1 + a_2 + \ldots a_n > b_1 + b_2 + \ldots b_n$ and $a_1 a_2 \ldots a_n > b_1 b_2 \ldots b_n$ if $b_1 b_2 \ldots b_n > 0$. In particular, if $a > b$ and n is a positive integer. Then $na > nb$ and $a^n > b^n$.
3. If $a > b$ then $-a < -b$ and $\frac{1}{a} < \frac{1}{b}$.
4. If $a > b$ and $x > 0$ then $ax > bx$.
5. If $a > b$ and $x < 0$, then $ax < bx$.
6. $a^2 > 0$ if $a \neq 0$ and $a^2 \geq 0$ for any real number a.

Solved Problems

1. If x, y, z and real but not all equal prove that $x^3 + y^3 + z^3 > 3xyz$ according as $x + y + z > 0$ and $x^3 + y^3 + z^3 < 3xyz$ according as $x + y + z < 0$.

Solution

$$x^3 + y^3 + z^3 = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

$$\therefore \sum x^3 = \frac{1}{2}\left(\pm\sum x\right)(2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2zx)$$

$$\text{ie } \sum x^3 = \frac{1}{2}\left(\sum x\right)\left[(x-y)^2 + (y-z)^2 + (z-x)^2\right] \quad (1)$$

The term within the bracket on RHS of (1) is always positive. ($\because$ x, y, z are not all equal). So $\sum x^3$ is +ve if $\sum x$ is +ve and $\sum x^3$ is –ve if $\sum x$ is –ve.

2. If a, b, c are the sides of a triangle. Prove that

$$\frac{1}{2} \leq \frac{ab + bc + ca}{a^2 + b^2 + c^3} \leq 1.$$

Solution

We know that

$$(a-b)^2 + (b-c)^2 + (c-a)^2 = 2\sum a^2 - 2\sum ab \quad (2)$$

$$\therefore 2\sum a^2 - 2\sum ab \geq 0$$

$$\text{ie } \sum a^2 \geq \sum ab$$

4. If a, b, c are positive integers, any two of which are together greater than the third; prove that.

$$\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c}>\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

Solution

$$\frac{1}{a+b-c}+\frac{1}{c+a-b}=\frac{c+a-b+a+b-c}{[a+(b-c)][a-(b-c)]}$$

$$=\frac{2a}{a^2-(b-c)^2}>\frac{2a}{a^2}$$

$$\therefore a^2-(b-c)^2<a^2$$

ie $$\frac{1}{a+b-c}+\frac{1}{c+a-b}>\frac{2}{a} \quad (7)$$

Similarly $$\frac{1}{b+c-a}+\frac{1}{a+b-c}>\frac{2}{b} \quad (8)$$

$$\frac{1}{c+a-b}+\frac{1}{b+c-a}>\frac{2}{c} \quad (9)$$

Adding (7), (8) and (9) and canceling 2 on both sides, we get the required in equality.

7.2.1 Arithmetic and Geometric Means

Definition

The arithmetic mean of $a_1, a_2, \ldots a_n$ is $\frac{a_1+a_2+\ldots+a_n}{n}$ and the geometric mean of $a_1, a_2, \ldots a_n$ is $(a_1 a_2 \ldots a_n)^{1/n}$

Theorm 1

The Arithmetic mean (A. M) of n positive numbers is greater than their geometric mean (given that the numbers are not all equal).

Proof

Take the numbers as $a_1, a_2, \ldots a_n$. First of all, we prove the theorem when $n=2^m$ where m any positive integer m by induction on m.

When $m=1, n=2$. In this case we have to prove that

$$\frac{a_1 + a_2}{2} > (a_1 a_2)^{1/2} \qquad (a_1 \neq a_2)$$

$$\left(\sqrt{a_1} - \sqrt{a_2}\right)^2 > 0$$

ie $\quad a_1 + a_2 - 2\sqrt{a_1 a_2} > 0$

Hence $\frac{a_1 + a_2}{2} > \sqrt{a_1 a_2}$

Thus there is basis for induction.

Assume the theorem for $n = 2^m$. We will prove the theorem for $n' = 2^{m+1} = 2_n$.

Let the numbers be $a_1, a_2, \ldots a_{2n}$

Then $A = \frac{a_1 + a_2 + \ldots + a_n}{n} > (a_1 a_2 \ldots a_n)^{1/n}$

and $A' = \frac{a_{n+1} + a_{n+2} + \ldots + a_{2n}}{n} > (a_{1+n} a_{n+2} \ldots a_{2n})^{1/n}$

We can assume that $A \neq A'$ by renaming $a_i's$ if necessary.

$$\frac{A + A'}{2} > (AA')^{1/2}$$

ie $\frac{\sum_{i=1}^{2n} a_i}{2n} \left[(a_1 a_2 \ldots a_n a_n + 1 \ldots a_{2n})^{1/n}\right]^{1/2}$

ie $\frac{\sum_{i=1}^{2n} a_i}{2n} > (a_1 a_2 \ldots a_{2n})^{1/2n}$

So the theorem is proved for $n' = 2^{m+1}$. By principle of induction the theorem is true for all n of the form 2^m.

We now prove the theorem for any n.

When n is not a power of 2, choose p such that $n + p = 2^m$ for some positive integer p.

Let $A = \frac{a_1 + a_2 \ldots a_n}{n}$

Consider the numbers $a_1, a_2, \ldots a_n, A, A \ldots$ (p times).

As we have proved the theorm for 2^m.

$$\therefore \frac{\sum ab}{\sum a^2} \le 1 \cdot$$

Hence the second inequality is proved.

Now $\dfrac{b^2+c^2-a^2}{2bc} = \cos A \le 1$

$$\therefore b^2+c^2-a^2 \le 2bc$$

Similarly $c^2+a^2-b^2 \le 2ca$

$$a^2+b^2+c^2 \le 2ab$$

Adding (3), (4) and (5), we get

$$a^2+b^2+c^2 < 2(ab+bc+ca)$$

So, $\dfrac{ab+bc+ca}{a^2+b^2+c^2} \ge \dfrac{1}{2}$

Hence the first inequality.

3. Prove that $(n!)^2 > n^n$ for $n > 2$

Solution

Let $1 < x < n$. Then $n - x > 0$ and $x - 1 > 0$

$$\therefore 0 < (n-x)(x-1) = -x^2 + (n+1)x - n = x(n+1-x) - n$$

Hence $\quad x(n+1-x) > n$ for $1 \; x < n$

Putting $x = 2,3,\ldots..n-1$ in (6), we get

$2(n-1) > n$

$3\,(n-2) > n$

.

.

$(n-1)\,(2) > n$

Multiplying the above in equalties we get

$$[(n-1)!]^2 > n^{n-2}$$

ie $(n!)^2 > n^{n-2}.n^2 = n^n$

$$\therefore (n!)^2 > n^n$$

$$\frac{1}{9}\left[\sum\left(\frac{1}{1-a}\right)\Big/_{3-a-b-c}\right]\geq 1$$

$$\therefore \quad \sum\left(\frac{1}{1-a}\right)\geq\frac{9}{2} \qquad (\text{since } 3-a-b-c=2)$$

7. If $s = x_1 + x_2 + ... + x_n$ so that

$$\frac{s}{s-x_1}+\frac{s}{s-x_2}+.....+\frac{s}{s-x_n}>\frac{n^2}{n-1} \tag{13}$$

Solution

Applying theorem 1 to $\frac{s}{s-x_1},\frac{s}{s-x_2},.....\frac{s}{s-x_n}$ and

$\frac{s-x_1}{s},\frac{s-x_2}{s},.....\frac{s-x_n}{s}$, we get

$$\frac{1}{n}\left[\sum_{i=1}^{n}\frac{s}{s-x_i}\right]\geq\left[\frac{s^n}{(s-x_1)(s-x_2).....(s-x_n)}\right]^{1/n} \tag{14}$$

$$\frac{1}{n}\left[\sum_{i=1}^{n}\frac{s}{s-x_i}\right]\geq\left[\frac{(s-x_1)(s-x_2).....(s-x_n)}{s^n}\right]^{1/n} \tag{15}$$

Multiplying in the equalities (14) and (15) we get

$$\frac{1}{n^2}\left[\sum_{i-1}^{n}\frac{s}{x-x_i}\right]\left[\frac{ns-x_1-x_2.....-x_n}{s}\right]\geq 1$$

ie $\quad \frac{1}{n^2}\left[\sum_{i-1}^{n}\frac{s}{x-x_i}\right]\left[\frac{(n-1)s}{s}\right]\geq 1$

Hence L.S.H. of (1) $\geq\frac{n^2}{n-1}$

8. Show that $\frac{2}{b+c}+\frac{2}{c+a}+\frac{2}{a+b}\geq\frac{9}{a+b+c}$

Solution

$$\frac{a_1+a_2+....+a_n+(A+A+.....p \text{ times})}{n+p} > \left[a_1a_2....a_nA^p\right]^{1/n+p}$$

ie $\frac{(n+p)A}{n+p} > \left[a_1a_2....a_nA^p\right]^{1/n+p}$

Raising both sides to the power $n+p$

$$A^{n+p} > a_1a_2....a_nA^p$$

ie $\quad A^{n+p} > a_1a_2....a_n$

ie $\quad A > (a_1a_2 \quad a_n)^{1/n}$

Thus the theorem is proved for any n.

Corollary

A.M. of n positive real numbers $\geq$ their G.M.

(When $a_1 = a_2 = ... a_n = a$, then A.M. = G.M. Hence the corollary)

Check Your Progress

1. Define an inequality.
2. What is the arithmetic mean of 5, 6, 7, 8, and 9?
3. What is the geometric mean of 1, 2, 3, 4, and 5?

7.3 WEIERSTRESS INEQUALITY

If a_1, a_2, a_n are positive numbers whose sum is s then

(a) $(1+a_1)(1+a_2)......(1+a_n) > 1+s$

(b) $(1-a_1)(1-a_2)......(1-a_n) > 1-s$ provided $a_i < 1$ for each i.

Proof

(a) We prove the result by induction on n.

When $n = 2$, $(1 + a_1)(1 + a_2) = 1 + a_1 + a_2 + a_1a_1 > 1 + a_1 + a_2$ since $a_1a_2 > 0$. So (a) is true when $n = 2$.

That is, $(1+a_1)(1+a_2).....(1+a_k) > 1+\sum_{i=1}^{k} a_i$ (10)

$$(1+a_1)(1+a_2).....(1+a_k)(1+a_{k+1})$$

Applying theorem 1, to $\frac{2}{b+c},\frac{2}{c+a},\frac{2}{a+b}$ and $\frac{b+c}{2},\frac{c+a}{2},\frac{a+b}{2}$ we get

$$\frac{1}{3}\left[\frac{2}{b+c}+\frac{2}{c+a}+\frac{2}{a+b}\right]\geq\left[\frac{8}{(b+c)(c+a)(a+b)}\right]^{1/3} \tag{16}$$

$$\frac{1}{3}\left[\frac{b+c}{2}+\frac{c+a}{2}+\frac{a+b}{2}\right]\geq\left[\frac{(b+c)(c+a)(a+b)}{8}\right]^{1/3} \tag{17}$$

Multiplying (16) and (17) we get

$$\frac{1}{9}\left[\frac{2}{b+c}+\frac{2}{c+a}+\frac{2}{a+b}\right][a+b+c]\geq 1$$

$\therefore$ L.H.S. of (1) $\geq \frac{9}{a+b+c}$, proving the required inequality.

9. If x and y are positive quantities whose sum is 4 show that

$$\left(x+\frac{1}{x}\right)^2\left(y+\frac{1}{y}\right)^2\geq\frac{25}{2}$$

Solution

$$\left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2=x^2+y^2+\frac{1}{x^2}+\frac{1}{y^2}+4 \tag{18}$$

$$\begin{aligned}\text{Now } x^2+y^2 &= x^2+(4-x)^2\\ &= 2x^2-8x+16\\ &= 2\left(x^2-4x+4\right)+8\\ &= 2(x-2)^2+8\end{aligned}$$

Hence $x^2+y^2\geq 8$ (19)

$$\frac{1}{2}\left(\frac{1}{x^2}+\frac{1}{y^2}\right)\geq\left(\frac{1}{x^2}\cdot\frac{1}{y^2}\right)^{1/2}$$

ie $$\frac{1}{x^2}+\frac{1}{y^2}\geq\frac{2}{xy} \tag{20}$$

$$>\left(1+\sum_{i=1}^{k} a_i\right)(1+a_{k+1}) \qquad \text{by (10)}$$

$$1+\sum_{i-1}^{k} a_i + a_{k+1} \text{ since } a_{k+1}\left(\sum_{i-1}^{k} a_i\right)>0.$$

So (a) is true for $k+1$.

By principle of induction the result (10) is true for all n.

Proof of (b) is similar.

Solved Problems

5. Show that $\frac{a_1}{a_2}+\frac{a_2}{a_3}+.....\frac{a_n}{a_1}>n$

Solution

By theorem 1,

$$\frac{1}{n}\left(\frac{a_1}{a_2}+\frac{a_2}{a_3}+....\frac{a_n}{a_1}\right)>\left(\frac{a_1}{a_2}.\frac{a_2}{a_3}.....\frac{a_n}{a_1}\right)^{1/n}$$

ie $$\frac{1}{n}\left(\frac{a_1}{a_2}+\frac{a_2}{a_3}+....\frac{a_n}{a_1}\right)>1$$

Hence the given in equality.

6. If $a+b+c=1$, show that $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}\geq\frac{9}{2}$

Solution

Applying theorem 1 to $\frac{1}{1-a},\frac{1}{1-b},\frac{1}{1-c}$ and $1-a$, $1-b$, $1-c$, we get

$$\frac{1}{3}\pm\sum\left(\frac{1}{1-a}\right)\geq\left[\frac{1}{(1-a)(1-b)(1-c)}\right] \tag{11}$$

$$\frac{1}{3}\pm\sum\left(\frac{1}{1-a}\right)\geq\left[(1-a)(1-b)(1-c)\right] \tag{12}$$

Multiplying the enequalities (11) and (12) (since $a>b$ and $c>d$, $a,b,c,d>0\Rightarrow ac>bd$) we get

Also, $\dfrac{x+y}{2} \geq (xy)^{1/2}$

ie $2 \geq \sqrt{xy}$ $\qquad \because (x+y=4)$

$\therefore 4 \geq xy$. So $\dfrac{2}{xy} > \dfrac{1}{2}$

Using this in (20) we get

$$\frac{1}{x^2}+\frac{1}{y^2} > \frac{1}{2} \tag{21}$$

From (18), (19) and (21)

$$\left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2 \geq 8+\frac{1}{2}+4$$

ie $\qquad \geq \dfrac{25}{2}$