The equation (1) can be written as

$$(x+1)(x-1)(6x^4-35x^3+62x^2-35x+6) = 0$$

i.e., $x+1 = 0;\quad x-1=0$

and $$6x^4-35x^3+62x^2-35x+6 = 0 \qquad \dots (2)$$

Equation (2) is a R.E. of Type I. Dividing by x^2, we get

$$6x^2-35x+62-\frac{35}{x}+\frac{6}{x^2} = 0$$

$$6\left(x^2+\frac{1}{x^2}\right)-35\left(x+\frac{1}{x}\right)+62 = 0 \qquad \dots (3)$$

Put $$x+\frac{1}{x} = u \qquad \dots (4)$$

$$\therefore x^2+\frac{1}{x^2} = u^2-2 \qquad \dots (5)$$

Substituting (4) and (5) in (3) we get

$$6(u^2-2)-35(u)+62 = 0$$
$$6u^2-35u+50 = 0$$
$$u = \frac{10}{3} \text{ or } \frac{5}{2}$$

i.e., $x+\dfrac{1}{x} = \dfrac{10}{3}$ or $x+\dfrac{1}{x} = \dfrac{5}{2}$

$3x^2-10x+3 = 0$ or $2x^2-5x+2 = 0$

$(x-3)(3x-1) = 0$ or $(x-2)(2x-1) = 0$

$x = 3 \text{ or } \dfrac{1}{3}$ or $x = 2 \text{ or } \dfrac{1}{2}$

$\therefore$ The roots of the given equation are $-1, 1, 3, \frac{1}{3}, 2, \frac{1}{2}$

Example 2 *Solve $6x^6-25x^5+31x^4-31x^2+25x-6=0$.* *[May 94, Madras]*

Solution

This is a reciprocal equation of even degree with unlike coefficients. Also the middle term is absent.

This is a Reciprocal Equation of Type IV.

$\therefore x = 1, -1$ are the roots of the given equation. Dividing the given equation by $(x-1)$ and $(x+1)$ we get

1	6	–25	31	0	–31	25	–6
	0	6	–19	12	12	–19	–6
–1	6	–19	12	12	–19	6	0
	0	–6	25	–37	25	–6	
	6	–25	37	–25	6	0	

$\therefore$ The given equation can be written as

$$(x-1)(x+1)(6x^4-25x^3+37x^2-25x+6) = 0$$

i.e., $x-1=0, x+1=0$

and $$6x^4-25x^3+37x^2-25x+6 = 0 \qquad \dots (1)$$

Dividing (1) by x^2, we get

$$6x^2-25x+37-\frac{25}{x}+\frac{6}{x^2} = 0$$

$$6\left(x^2+\frac{1}{x^2}\right)-25\left(x+\frac{1}{x}\right)+37 = 0 \qquad \dots (2)$$

Put $x+\dfrac{1}{x} = u,$ $$x^2+\frac{1}{x^2} = u^2-2 \qquad \dots (3)$$

Substituting (3) in (2) we get

$$6(u^2-2)-25u+37 = 0$$
$$6u^2-25u+25 = 0$$
$$u = \frac{25\pm\sqrt{625-600}}{12}$$
$$= \frac{25\pm5}{12} = \frac{30}{12} \text{ or } \frac{20}{12}$$
$$u = \frac{5}{2} \text{ or } \frac{5}{3}$$

i.e., $x+\dfrac{1}{x} = \dfrac{5}{2}$ or $x+\dfrac{1}{x} = \dfrac{5}{3}$

$x^2-\dfrac{5}{2}x+1 = 0$ or $x^2-\dfrac{5}{3}x+1 = 0$

$2x^2-5x+2 = 0$ or $3x^2-5x+3 = 0$

$x = 2 \text{ or } \dfrac{1}{2}$ $\qquad x = \dfrac{5\pm i\sqrt{11}}{6}$

$\therefore$ The roots of the given equation are $1, -1, 2, \frac{1}{2}, \frac{5\pm i\sqrt{11}}{6}$

Exercises

I. Solve the following reciprocal equations.

1. $6x^4-25x^3+37x^2-25x+6=0.$ *[Ans. $2, \frac{1}{2}, \frac{5\pm i\sqrt{11}}{6}$]*
2. $6x^4-35x^3+62x-35x+6=0.$ *[Ans. $3, \frac{1}{3}, 2, \frac{1}{2}$]*
3. $4x^4-20x^3+33x^2-20x+4=0.$ *[Ans. $2, \frac{1}{2}, 2, \frac{1}{2}$]*

III 1. Solve the equation $6x^3 - 11x^2 + 6x - 1 = 0$ whose roots in H.P.

[Ans. $1, \frac{1}{2}, \frac{1}{2}$]

2. Solve the equation $81x^3 - 18x^2 - 36x + 8 = 0$ whose roots are in H.P. [Ans. $\frac{2}{9}, \frac{2}{3}, \frac{-2}{3}$]
3. If the roots of $x^3 + 3ax^2 + 3bx + c = 0$ are in H.P., show that $2b^3 = c\,(3ab - c)$.
4. If the roots of the equation $x^3 - ax^2 + bx - c = 0$ are in H.P. show that the mean root is $\frac{3c}{b}$.

   [Hint : If the roots are $\alpha - d, \alpha, \alpha + d$, then the mean root is α]
5. Solve the equation $40x^4 + 22x^3 - 21x^2 - 2x + 1 = 0$ given that the roots are in H.P. [Ans. $\frac{-1}{4}, -1, \frac{1}{2}, \frac{1}{5}$]

To increase or decrease the roots of a given equation by a given quantity

Increasing and decreasing the roots of a given equation by a given quantity is exihibited in the following examples.

Note : To increase the roots of an equation by h, diminish the roots of that equation by $-h$.

Example 1 *Diminish the roots of $x^4 - 5x^3 + 7x^2 - 4x + 5 = 0$ by 2 and find the transformed equation.*

Solution

Diminishing the roots by 2, we get

2	1	–5	7	–4	5	
	0	2	–6	2	–4	
2	1	–3	1	–2	1	(absolute term of the transformed equation)
	0	2	–2	–2		
2	1	–1	–1	–4		(coefficient of x)
	0	2	2			
2	1	1	1			(coefficient of x^2)
	0	2				
2	1	3				(coefficient of x^3)
	0					
	1					(coeff. of x^4 in the transformed equation)

The transformed equation whose roots are less by 2 of the given equation is $x^4 + 3x^3 + x^2 - 4x + 1 = 0$

Example 2 *Diminish the roots of $2x^5 - x^3 + 10x - 8 = 0$ by 5 and find the transformed equation.*

Solution

Diminishing the roots by 2, we get

5	2	0	–1	0	10	–8	
	0	10	50	245	1225	6175	
5	2	10	49	245	1235	6167	(constant term of of transformed equation)
	0	10	100	745	4950		
5	2	20	149	–2	6185		(coefficient of x)
	0	10	150	1495			
5	2	30	299	2485			(coefficient of x^2)
	0	10	200				
5	2	40	499				(coefficient of x^3)
	0	10					
5	2	50					(coefficient of x^4)
	0						
	2						(coeff. of x^5 in the transformed equation)

The transformed equation is

$$2x^5 + 50x^4 + 499x^3 + 2485x^2 + 6185x + 6167 = 0$$

Example 3 *Increase by 7 the roots of the equation $3x^4 + 7x^3 - 15x^2 + x - 2 = 0$ and find the transformed equation.*

Solution

Increasing by 7 the roots of the given equation is the same as diminishing the roots by –7.

–7	3	7	–15	1	–2	
	0	–21	98	–581	406	
					0	
–7	3	–14	83	–580	405 8	(absolute term)
	0	–21	245	–2296		
–7	3	–35	328	–2876		(coefficient of x)
	0	–21	392			
–7	3	–56	720			(coefficient of x^2)
	0	–21				
–7	3	–77				(coefficient of x^3)
	0					
	3					(coeff. of x^4 in the transformed equation)

The transformed equation is

$$3x^4 - 77x^3 + 720x^2 - 2876x + 4058 = 0$$

$$6x^2+5x-38+\frac{5}{x}+\frac{6}{x^2} = 0$$

i.e., $$6\left(x^2+\frac{1}{x^2}\right)+5\left(x+\frac{1}{x}\right)-38 = 0 \qquad \ldots (3)$$

Put $$x+\frac{1}{x} = u \qquad \ldots (4)$$

$$\therefore x^2+\frac{1}{x^2} = u^2-2 \qquad \ldots (5)$$

Substituting (4) and (5) in (3) we get

$$6(u^2-2)+5u-38 = 0$$
$$6u^2+5u-50 = 0$$

i.e., $$u = \frac{5}{2} \text{ or } \frac{-10}{3}$$

i.e., $$x+\frac{1}{x} = \frac{5}{2} \quad \text{or} \quad x+\frac{1}{x} = \frac{-10}{3}$$

$$2x^2-5x+2 = 0 \quad \text{or} \quad 3x^2+10x+3 = 0$$

Solving these two quadratic equation we get

$$x = \frac{1}{2} \text{ or } 2 \quad \text{or} \quad x = \frac{-1}{3} \text{ or } -3$$

$\therefore$ The roots of the given equation are $1, \frac{1}{2}, 2, \frac{-1}{3}, -3$.

Example 2 *Solve $x^5-5x^4+9x^3-9x^2+5x-1=0$.*

Solution

Given $$x^5-5x^4+9x^3-9x^2+5x-1 = 0 \qquad \ldots (1)$$

This is a Reciprocal Equation of odd degree with unlike signs.

i.e., Reciprocal Equation of Type III.

$\therefore$ $x=1$ is a root of (1).

1	1	−5	9	−9	5	−1
	0	1	−4	5	−4	1
	1	−4	5	−4	1	0

Equation (1) can be written as

$$(x-1)(x^4-4x^3+5x^2-4x+1) = 0$$

i.e., $$x-1 = 0$$

or $$x^4-4x^3+5x^2-4x+1 = 0 \qquad \ldots (2)$$

Dividing (2) by x^2, we get

$$x^2-4x+5-\frac{4}{x}+\frac{1}{x^2} = 0$$

$$\left(x^2+\frac{1}{x^2}\right)-4\left(x+\frac{1}{x}\right)+5 = 0 \qquad \ldots (3)$$

Put $$x+\frac{1}{x} = u \qquad \ldots (4)$$

$$x^2+\frac{1}{x^2} = u^2-2 \qquad \ldots (5)$$

Substituting (4) and (5) in (3) we get

$$(u^2-2)-4u+5 = 0$$
$$u^2-4u+3 = 0$$
$$u = 3 \text{ or } 1$$

i.e., $$x+\frac{1}{x} = 3 \quad \text{or} \quad x+\frac{1}{x} = 1$$

$$x^2-3x+1 = 0 \quad \text{or} \quad x^2-x+1 = 0$$

Solving these two equations we get

$$x = \frac{3\pm\sqrt{9-4}}{2} \quad \text{or} \quad x = \frac{1\pm\sqrt{1-4}}{2}$$

$$x = \frac{3\pm\sqrt{5}}{2} \quad \text{or} \quad x = \frac{1\pm\sqrt{3}\,i}{2}$$

$\therefore$ The roots of the given equation are $1, \frac{3\pm\sqrt{5}}{2}, \frac{1\pm\sqrt{3}\,i}{2}$.

Type IV : Reciprocal equations with even degree and the terms equidistant from the first and last have opposite signs and the middle term is absent. For this type $x=1$ and $x=-1$ are the roots of the given Reciprocal Equation. Now dividing the given equation by $(x-1)$ $(x+1)$ we get a Reciprocal Equation of degree 4 which can be solved by using Type I.

Example 1 *Solve the equation*

$6x^6-35x^5+56x^4-56x^2+35x-6=0$.

Solution

Given $$6x^6-35x^5+56x^4-56x^2+35x-6 = 0 \qquad \ldots (1)$$

This is a reciprocal equation of even degree with unlike signs. *i.e.,* Reciprocal equation of Type IV.

$\therefore$ $x=1$ and -1 are the roots of (1). Since x^3 term is absent its coefficient is zero.

1	6	−35	56	0	−56	35	−6
	0	6	−29	27	27	−29	6
−1	6	−29	27	27	−29	6	0
	0	−6	35	−62	35	−6	
	6	−35	62	−35	6	0	

$$a_0\left(\frac{y}{m}\right)^n + a_1\left(\frac{y}{m}\right)^{n-1} + a_2\left(\frac{y}{m}\right)^{n-2} + \ldots + a_n = 0 \qquad \ldots (2)$$

Multiplying (2) by m^n, we get

$$a_0y^n + ma_1y^{n-1} + m^2a_2y^{n-2} + \ldots + m^na_n = 0$$

i.e., $$a_0x^n + ma_1x^{n-1} + m^2a_2x^{n-2} + \ldots + m^na_n = 0$$

(changing y in to x).

Thus to transform an equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$$

into another equation whose roots are m times those of the given equation, then just multiply the successive coefficients beginning with the second by $m, m^2, m^3, \ldots$, etc.

Note : This transformation is useful for the purpose of removing the coefficient of the first term of an equation when it is other than unity and generally for removing the fractional coefficients from an equation.

Example 1 ***Multiply the roots of the equation*** $x^4 + 2x^3 + 4x^2 + 6x + 8 = 0$ ***by*** $\frac{1}{2}$.

Solution

Given $$x^4 + 2x^3 + 4x^2 + 6x + 8 = 0 \qquad \ldots (1)$$

To multiply the roots of (1) by $\frac{1}{2}$, we have to multiply the succes- sive coefficients beginning with the second by $\frac{1}{2}, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \left(\frac{1}{2}\right)^4$

i.e., $$x^4 + \left(\frac{1}{2}\right)2x^3 + \left(\frac{1}{2}\right)^2 4x^2 + \left(\frac{1}{2}\right)^3 6x + \left(\frac{1}{2}\right)^4 8 = 0$$

i.e., $$x^4 + x^3 + x^2 + \frac{3}{4}x + \frac{1}{2} = 0$$

i.e., $$4x^4 + 4x^3 + 4x^2 + 3x + 2 = 0$$

which is the required equation.

Example 2 ***Multiply the roots of*** $x^3 - 3x + 1 = 0$ ***by 10.***

Solution

The given equation can be written as

$$x^3 + 0x^2 - 3x + 1 = 0 \qquad \ldots (1)$$

Multiply the roots of (1) by 10 we get

$$x^3 + (10)\cdot 0x^2 - (10)^2\cdot 3x + (10)^3\cdot 1 = 0$$

i.e., $$x^3 - 300x + 1000 = 0$$

which is the required equation.

Example 3 ***Transform the equation*** $3x^3 + 4x^2 + 5x - 6 = 0$ ***into one in which the coefficient of*** x^3 ***is unity.***

Solution

Given $$3x^3 + 4x^2 + 5x - 6 = 0 \qquad \ldots (1)$$

Multiply the roots of (1) by 3, we get

$$3x^3 + 3\cdot 4x^2 + 3^2\cdot 5x - 3^3\cdot 6 = 0$$

Dividing by 3, we get $$x^3 + 4x^2 + 15x - 54 = 0$$

Example 4 ***Remove the fractional coefficients from the equation*** $x^3 - \frac{1}{4}x^2 + \frac{1}{3}x - 1 = 0.$

Solution

Given $$x^3 - \frac{1}{4}x^2 + \frac{1}{3}x - 1 = 0 \qquad \ldots (1)$$

Multiply the roots of (1) by m, we get

$$x^3 - \frac{m}{4}x^2 + \frac{m^2}{3}x - m^3 = 0 \qquad \ldots (2)$$

If $m = 12$ (L.C.M. of 4 and 3), the fractions will be removed. Put $m = 12$ in (2), we get

$$x^3 - 3x^2 + 48x - 1728 = 0$$

To transform an equation into another equation whose roots ar[e] the reciprocals of the roots of the given equation.

Let $$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0 \qquad \ldots (1)$$

be the given equation.

If x is the root of (1), then $y = \frac{1}{x}$ is the root of the required equation.

Substituting $x = \frac{1}{y}$ in (1) we get

$$\frac{a_0}{y^n} + \frac{a_1}{y^{n-1}} + \frac{a_2}{y^{n-2}} + \ldots + \frac{a_{n-1}}{y} + a_n = 0$$

Multiplying by y^n, we get

$$a_ny^n + a_{n-1}y^{n-1} + \ldots + a_1y + a_0 = 0$$

or $$a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = 0$$

which is the required equation whose roots are the reciprocals of th[e] of the given equation.

To transform an equation of the nth degree into another roots are reciprocals of the roots of the given equation, change in the given equation and multiply the resulting equation by x^n

$$u = 1 \text{ or } -4$$

i.e., $x + \frac{1}{x} = 1$, or $x + \frac{1}{x} = -4$.

$$x^2 - x + 1 = 0 \quad \text{or} \quad x^2 + 4x + 1 = 0$$

$$x = \frac{1 \pm \sqrt{1-4}}{2} \quad \text{or} \quad x = \frac{-4 \pm \sqrt{16-4}}{2}$$

$$= \frac{1 \pm \sqrt{3}\, i}{2} \quad \text{or} \quad = \frac{-4 \pm 2\sqrt{3}}{2}$$

$$x = \frac{1 \pm \sqrt{3}\, i}{2} \quad \text{or} \quad x = -2 \pm \sqrt{3}$$

∴ The roots of the given equation are $-1, \frac{1 \pm \sqrt{3} i}{2}, -2 \pm \sqrt{3}$

Example 2 *Solve $6x^5 + 11x^4 - 33x^3 - 33x^2 + 11x + 6 = 0$.*

Solution

Given

$$6x^5 + 11x^4 - 33x^3 - 33x^2 + 11x + 6 = 0 \quad \dots (1)$$

This is a Reciprocal Equation of Type II.

∴ $x = -1$ is a root of (1).

−1	6	11	−33	−33	11	6
	0	−6	−5	38	−5	−6
	6	5	−38	5	6	0

∴ The given equation can be written as

$$(x+1)(6x^4 + 5x^3 - 38x^2 + 5x + 6) = 0$$

i.e.,
$$x + 1 = 0$$

(or)
$$6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0 \quad \dots (2)$$

This is a Reciprocal Equation of Type I.

Dividing (2) by x^2, we get

$$6x^2 + 5x - 38 + \frac{5}{x} + \frac{6}{x^2} = 0$$

$$6\left(x^2 + \frac{1}{x^2}\right) + 5\left(x + \frac{1}{x}\right) - 38 = 0 \quad \dots (3)$$

Put
$$x + \frac{1}{x} = u \quad \dots (4)$$

$$\therefore x^2 + \frac{1}{x^2} = u^2 - 2 \quad \dots (5)$$

Substituting (4) and (5) in (3) we get

$$6(u^2 - 2) + 5u - 38 = 0$$

$$6u^2 + 5u - 50 = 0$$

$$u = \frac{-5 \pm \sqrt{25 + 1200}}{12} = \frac{-5 \pm 35}{12}$$

$$u = \frac{30}{12} \text{ or } \frac{-40}{12}$$

$$= \frac{5}{2} \text{ or } \frac{-10}{3}$$

i.e., $x + \frac{1}{x} = \frac{5}{2}$ or $x + \frac{1}{x} = \frac{-10}{3}$

$$x^2 - \frac{5}{2}x + 1 = 0 \quad \text{or} \quad x^2 + \frac{10}{3}x + 1 = 0$$

$$2x^2 - 5x + 2 = 0 \quad \text{or} \quad 3x^2 + 10x + 3 = 0$$

Solving we get $2, \frac{1}{2}$ and $-3, \frac{-1}{3}$

∴ The roots of the given equation are $-1, 2, \frac{1}{2}, -3, \frac{-1}{3}$.

Type III : Reciprocal equations of odd degree with unlike signs for its coefficients

For this type $x = 1$ is a root or $(x - 1)$ is a factor of the given Reciprocal Equation. Now dividing the given Reciprocal Equation by $x - 1$, we get a Reciprocal Equation of degree four which can be solved by using Type I.

Example 1 *Solve the equation $6x^5 - x^4 - 43x^3 + 43x^2 + x - 6 = 0$.*

Solution

Given $6x^5 - x^4 - 43x^3 + 43x^2 + x - 6 = 0 \quad \dots (1)$

This is a Reciprocal Equation of odd degree with unlike signs.

i.e., Reciprocal equatiion of Type III.

∴ $x = 1$ is a root of (1).

1	6	−1	−43	43	1	−6
	0	6	5	−38	5	6
	6	5	−38	5	6	0

Equation (1) can be written as

$$(x-1)(6x^4 + 5x^3 - 38x^2 + 5x + 6) = 0$$

$$x - 1 = 0$$

or
$$6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0 \quad \dots (2)$$

This is a Reciprocal Equation of Type I.

Dividing (2) by x^2, we get

Example 1 *If $\alpha, \beta, \gamma, \delta$ are the roots of $x^4+px^3+qx^2+rx+s=0$, find the equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$.*

Solution

Given $$x^4+px^3+qx^2+rx+s = 0 \quad \ldots (1)$$

Its roots are α, β, γ and δ.

To find the equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$ (reciprocals of (1))

change x to $\frac{1}{x}$ in (1), we get

$$\left(\frac{1}{x}\right)^4 + p\left(\frac{1}{x}\right)^3 + q\left(\frac{1}{x}\right)^2 + r\left(\frac{1}{x}\right) + s = 0$$

i.e., $$sx^4+rx^3+qx^2+px+1 = 0$$

which is the required equation.

Example 2 *If 1, 2, 3, 6 are the roots of the equation $x^4-12x^3+47x^2-72x+36=0$, find an equation whose roots are $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}$.*

Solution

Given $$x^4-12x^3+47x^2-72x+36 = 0 \quad \ldots (1)$$

Its roots are 1, 2, 3, 6. Change x to $\frac{1}{x}$ in (1), we get

$$\left(\frac{1}{x}\right)^4 - 12\left(\frac{1}{x}\right)^3 + 47\left(\frac{1}{x}\right)^2 - 72\left(\frac{1}{x}\right) + 36 = 0$$

i.e., $$36x^4-72x^3+47x^2-12x+1 = 0$$

which is the required equation whose roots are $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}$.

Example 3 *Solve the equation $6x^3-11x^2-3x+2=0$ given that its roots are in H.P.*

Solution

Given $$6x^3-11x^2-3x+2 = 0 \quad \ldots (1)$$

Its roots are in H.P. Change x to $\frac{1}{x}$ in (1), we get

$$6\left(\frac{1}{x}\right)^3 - 11\left(\frac{1}{x}\right)^2 - 3\left(\frac{1}{x}\right) + 2 = 0$$

i.e., $$2x^3-3x^2-11x+6 = 0 \quad \ldots (2)$$

Now the roots of (2) are in A.P. [$\because$ H.P. is a reciprocal of A.P.] Let the roots of (2) be $\alpha-d, \alpha, \alpha+d$.

Sum of the roots
$$\alpha-d+\alpha+\alpha+d = \frac{3}{2}$$
$$3\alpha = \frac{3}{2}$$
$$\alpha = \frac{1}{2} \quad \ldots (3)$$

Product of the roots taken 3 at a time is
$$(\alpha-d)\,\alpha\,(\alpha+d) = -3$$
$$\alpha(\alpha^2-d^2) = -3$$
$$\frac{1}{2}\left(\frac{1}{4}-d^2\right) = -3$$
$$\frac{1}{8}-\frac{1}{2}d^2 = -3$$
$$3+\frac{1}{8} = \frac{1}{2}d^2$$
i.e., $$d^2 = \frac{50}{8}$$
$$d = \pm\sqrt{\frac{50}{8}} = \pm\frac{\sqrt{2}\cdot 5}{\sqrt{2}\cdot 2}$$
$$d = \pm\frac{5}{2}$$

Case (i) : When $d=\frac{5}{2}$ and $\alpha=\frac{1}{2}$, the roots of (2) are $\frac{1}{2}-\frac{5}{2}, \frac{1}{2}, \frac{1}{2}+\frac{5}{2}$ *i.e.,* $-2, \frac{1}{2}, 3$.

$\therefore$ The roots of the given equation are the reciprocal of the roots of (2).

i.e., $\frac{-1}{2}, 2, \frac{1}{3}$ are the roots of (1).

Case (ii) : When $d=\frac{-5}{2}$ and $\alpha=\frac{1}{2}$ the roots of (2) are $\frac{1}{2}+\frac{5}{2}, \frac{1}{2}, \frac{1}{2}-\frac{5}{2}$ *i.e.,* $3, \frac{1}{2}, -2$.

The reciprocal of these roots viz. $\frac{1}{3}, 2, \frac{-1}{2}$ are the roots of the given equation.

Example 4 *Solve the equation $15x^4-8x^3-14x^2+8x-1=0$ given that the roots are in H.P.*

Solution

Given $$15x^4-8x^3-14x^2+8x-1 = 0 \quad \ldots (1)$$

Example 4 *Find the equation whose roots are the roots of $x^4 - x^3 - 10x^2 + 4x + 24 = 0$ increased by 2.*

Solution

Increasing by 2 the roots of the given equation is the same as diminishing the roots by -2.

-2	1	-1	-10	4	24
	0	-2	6	8	-24
-2	1	-3	-4	12	0
	0	-2	10	-12	
-2	1	-5	6	0 (coefficient of x in the transformed equation)	
	0	-2	14		
-2	1	-7	20 (coefficient of x^2)		
	0	-2			
-2	1	-9 (coefficient of x^3)			
	0				
	1 (coeff. of x^4 in the transformed equation)				

The transformed equation is $x^4 - 9x^3 + 20x = 0$

Example 5 *If α, β, γ are the roots of the equation $x^3 - 6x^2 + 12x - 8 = 0$, find an equation whose roots are $\alpha - 2, \beta - 2, \gamma - 2$.*

Solution

Here we have to diminish the roots of $x^3 - 6x^2 + 12x - 8 = 0$ by 2.

Diminish the roots by 2, we get

2	1	-6	12	-8
	0	2	-8	8
2	1	-4	4	0
	0	2	-4	
2	1	-2	0	
	0	2		
	1	0		

The transformed equation is $x^3 = 0$

i.e., the roots are $= 0, 0, 0$

i.e., $\alpha - 2 = 0;\ \beta - 2 = 0,\ \gamma - 2 = 0.$

i.e., $\alpha = 2,\ \beta = 2,\ \gamma = 2$

Example 6 *If α, β, γ are the roots of $8x^3 - 4x^2 + 6x - 1 = 0$, find the equation whose roots are $\alpha + \frac{1}{2}; \beta + \frac{1}{2}, \gamma + \frac{1}{2}$.*

Solution

Here we have to increase the roots of the given equation by $\frac{1}{2}$ *i.e.,* diminish the roots of the given equation by $\frac{-1}{2}$.

$-\frac{1}{2}$	8	-4	6	-1
	0	-4	4	-5
$-\frac{1}{2}$	8	-8	10	-6
	0	-4	6	
$-\frac{1}{2}$	8	-12	16	
	0	-4		
$-\frac{1}{2}$	8	-16		
	0			
	8			

The equation whose roots are $\alpha + \frac{1}{2}, \beta + \frac{1}{2}, \gamma + \frac{1}{2}$ is

$$8x^3 - 16x^2 + 16x - 6 = 0$$

Exercises

I. Diminish the roots of the following equations and find the transformed equation.

1. $x^5 - 4x^4 + 3x^3 - 4x + 6 = 0$ by 3. *[Ans. $x^5 + 11x^4 + 42x^3 + 57x^2 - 13x - 60 = 0$]*
2. $x^4 - 5x^3 + 7x^2 - 4x + 5 = 0$ by 3. *[Ans. $x^4 + 3x^3 + x^2 - 4x + 1 = 0$]*
3. $x^4 - 5x^3 + 7x^2 - 17x + 11 = 0$ by 2. *[Ans. $x^4 + 3x^3 + x^2 - 17x - 19 = 0$]*
4. $x^4 - 6x^3 - x^2 + 30x - 20 = 0$ by 3. *[Ans. $x^4 + 6x^3 - x^2 - 30x - 20 = 0$]*

II. Increase the roots of the following equations and find the transformed equation.

1. $x^3 - 4x^2 + 3x - 1 = 0$ by 2. *[Ans. $x^3 - 10x^2 + 31x - 31 = 0$]*
2. $x^4 - x^3 - 10x^2 + 4x + 24 = 0$ by 2 and hence solve it. *[Ans. $-2, -2, 2, 3$]*
3. $4x^4 + 32x^3 + 83x^2 + 76x + 21 = 0$ by 2 and hence solve it. *[Ans. $\frac{1}{2}, \frac{7}{2}, -1, -3$]*
4. $3x^5 - 5x^3 + 7 = 0$ by 4. *[Ans. $3x^5 - 60x^4 + 475x^3 - 1860x^2 + 3600x - 2745 = 0$]*

❑ RECIPROCAL EQUATIONS

Definition : If an equation remains unaltered when x is changed into $\frac{1}{x}$ (reciprocal of x), then it is called a ***reciprocal equation.***

Let $$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n = 0 \quad \dots (1)$$

be a reciprocal equation. When x is changed into $\frac{1}{x}$, we get

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x + 1 = 0$$

i.e., $$x^n + \frac{a_{n-1}}{a_n} x^{n-1} + \frac{a_{n-2}}{a_n} x^{n-2} + \dots + \frac{a_1}{a_n} x + \frac{1}{a_n} = 0 \quad \dots (2)$$

Since (1) is a reciprocal equation, it is same as (2).

$\therefore$ Equating like coefficients in (1) and (2), we get

$$\frac{a_{n-1}}{a_n} = a_1, \frac{a_{n-2}}{a_n} = a_2, \dots, \frac{a_1}{a_n} = a_{n-1} \text{ and } \frac{1}{a_n} = a_n.$$

i.e., $$a_n^2 = 1$$
$$a_n = \pm 1$$

Case (i) : When $a_n = 1$.

Then $a_{n-1} = a_1, a_{n-2} = a_2, a_{n-3} = a_3, \dots$

In this case the coefficients of the terms equidistant from the beginning and the end are equal in magnitude and have the same sign.

Case (ii) : When $a_n = -1$, we have

$a_{n-1} = -a_1, a_{n-2} = -a_2, \dots, a_1 = -a_{n-1}$.

In this case the terms equidistant from the beginning and the end are equal in magnitude but different in sign.

Note : If α be a root of a reciprocal equation, $\frac{1}{\alpha}$ must also be a root. Hence the roots of a reciprocal equation occur in pairs like $\alpha, \frac{1}{\alpha}, \beta, \frac{1}{\beta}, \dots$

STANDARD FORMS OF RECIPROCAL EQUATIONS

Type I : Reciprocal equation of degree four with like and unlike signs for its coefficients.

Type II : Reciprocal equation of odd degree with like signs for its coefficients. In this case $x = -1$ is a root of the given equation.

Type III : Reciprocal equation of odd degree with unlike signs for its coefficients. In this case $x = 1$ is a root of the given equation.

Type IV : Reciprocal equation of even degree with unlike signs for its coefficients and the middle term is absent. In this case $x = 1, -1$ are the roots of the given equation.

The following examples explain the methods of solving the above types of reciprocal equations.

Note 1 : A reciprocal equation with like signs for its coefficients means the terms equidistant from the beginning and the end are equal in magnitude and with same sign.

E.g., $$x^5 + 4x^4 + x^3 + x^2 + 4x + 1 = 0$$

Note 2 : A reciprocal equation with unlike signs for its coefficients means the terms equidistant from the beginning and the end are equal in magnitude but opposite in sign.

E.g., $$6x^6 + 35x^5 + 56x^4 + 56x^2 + 35x - 6 = 0$$

Example 1 ***Solve $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$.***

Solution

Given $$x^4 - 10x^3 + 26x^2 - 10x + 1 = 0 \quad \dots (1)$$

This is a reciprocal equation of Type I.

Dividing (1) by x^2, we get,

$$x^2 - 10x + 26 - \frac{10}{x} + \frac{1}{x^2} = 0$$

i.e., $$\left(x^2 + \frac{1}{x^2}\right) - 10\left(x + \frac{1}{x}\right) + 26 = 0 \quad \dots (2)$$

$$\text{Put } x + \frac{1}{x} = u \quad \dots (3)$$

$$\therefore x^2 + \frac{1}{x^2} = u^2 - 2 \quad \dots (4)$$

Substituting (3) and (4) in (2) we get

$$(u^2 - 2) - 10u + 26 = 0$$
$$u^2 - 10u + 24 = 0$$

which is a quadratic equation in u.

$$\text{i.e.,} \quad (u-6)(u-4) = 0$$

$$\text{i.e.,} \quad u = 4 \quad \text{or} \quad u = 6$$

$$\text{i.e.,} \quad x + \frac{1}{x} = 4 \quad \text{or} \quad x + \frac{1}{x} = 6$$

$$\text{i.e.,} \quad x^2 - 4x + 1 = 0 \quad \text{or} \quad x^2 - 6x + 1 = 0$$

$$x = \frac{4 \pm \sqrt{16-4}}{2} \qquad x = \frac{6 \pm \sqrt{36-4}}{2}$$

$$= \frac{4 \pm 2\sqrt{3}}{2} \qquad = \frac{6 \pm 4\sqrt{2}}{2}$$

$$\text{i.e.,} \quad x = 2 \pm \sqrt{3} \quad \text{or} \quad x = 3 \pm 2\sqrt{2}$$

$\therefore$ The roots of the given equation are

$2+\sqrt{3},\ 2-\sqrt{3},\ 3+2\sqrt{2},\ 3-2\sqrt{2}$

Example 2 *Solve $4x^4 - 20x^3 + 33x^2 - 20x + 4 = 0$.*

Solution

Given $\quad 4x^4 - 20x^3 + 33x^2 - 20x + 4 = 0 \quad \ldots (1)$

This is a reciprocal equation of Type I.

Dividing (1) by x^2, we get,

$$4x^2 - 20x + 33 - \frac{20}{x} + \frac{4}{x^2} = 0$$

$$4\left(x^2 + \frac{1}{x^2}\right) - 20\left(x + \frac{1}{x}\right) + 33 = 0 \quad \ldots (2)$$

Put $\quad x + \frac{1}{x} = u \quad \ldots (3)$

$$\therefore x^2 + \frac{1}{x^2} = u^2 - 2 \quad \ldots (4)$$

Substituting (3) and (4) in (2) we get

$$4(u^2 - 2) - 20u + 33 = 0$$

$$4u^2 - 20u + 25 = 0$$

$$u = \frac{20 \pm \sqrt{400-400}}{8}$$

$$\text{i.e.,} \quad u = \frac{5}{2} \text{ or } u = \frac{5}{2}$$

$$\text{i.e.,} \quad x + \frac{1}{x} = \frac{5}{2}$$

$$x^2 - \frac{5}{2}x + 1 = 0$$

$$\text{i.e.,} \quad 2x^2 - 5x + 2 = 0$$

This is a quadratic equation in x. $\quad \therefore x = \frac{5 \pm \sqrt{25-16}}{4} = \frac{5 \pm 3}{4}$

$$\text{i.e.,} \quad x = 2 \text{ or } \frac{1}{2}$$

$\therefore$ The roots of the given equation are $2, \frac{1}{2}, 2, \frac{1}{2}$.

Type II : Reciprocal equations of odd degree with like signs for its coefficients

For this type $x = -1$ is a root of the given Reciprocal Equation. Now dividing the given Reciprocal Equation by $x + 1$ we get a Reciprocal Equation of degree four which is clearly of Type I.

Example 1 *Solve $x^5 + 4x^4 + x^3 + x^2 + 4x + 1 = 0$.*

Solution

Given $\quad x^5 + 4x^4 + x^3 + x^2 + 4x + 1 = 0 \quad \ldots (1)$

This is a Reciprocal Equation of Type II.

$\therefore x = -1$ is a root of (1)

-1	1	4	1	1	4	1
	0	-1	-3	2	-3	-1
	1	3	-2	3	1	0

$\therefore$ The equation can be written as

$$(x+1)(x^4 + 3x^3 - 2x^2 + 3x + 1) = 0$$

$$x + 1 = 0 \text{ or } x^4 + 3x^3 - 2x^2 + 3x + 1 = 0 \quad \ldots (2)$$

This is a Reciprocal Equation of Type I

Dividing (2) by x^2, we get

$$x^2 + 3x - 2 + \frac{3}{x} + \frac{1}{x^2} = 0$$

$$\left(x^2 + \frac{1}{x^2}\right) + 3\left(x + \frac{1}{x}\right) - 2 = 0 \quad \ldots (3)$$

Put $x + \frac{1}{x} = u$, $\quad \therefore x^2 + \frac{1}{x^2} = u^2 - 2 \quad \ldots (4)$

Substituting (4) in (3) we get,

$$u^2 - 2 + 3u - 2 = 0$$

$$u^2 + 3u - 4 = 0$$

$$u = \frac{-3 \pm \sqrt{9+16}}{2} = \frac{-3 \pm 5}{2}$$

Its roots are in H.P. Change x into $\frac{1}{x}$ in (1) we get,

$$15\left(\frac{1}{x^4}\right) - 8\left(\frac{1}{x^3}\right) - 14\left(\frac{1}{x^2}\right) + 8\left(\frac{1}{x}\right) - 1 = 0$$

i.e., $$-x^4 + 8x^3 - 14x^2 - 8x + 15 = 0$$

i.e., $$x^4 - 8x^3 + 14x^2 + 8x - 15 = 0 \quad \ldots (2)$$

Its roots are in A.P. Let the roots be $\alpha - 3d, \alpha - d, \alpha + d, \alpha + 3d$.

Sum of the roots $= \alpha - 3d + \alpha - d + \alpha + d + \alpha + 3d = 8$

i.e., $$4\alpha = 8$$
$$\alpha = 2$$

Sum of the products of the roots taken four at a time is

$$(\alpha - 3d)(\alpha - d)(\alpha + d)(\alpha + 3d) = -15$$
$$(\alpha^2 - 9d^2)(\alpha^2 - d^2) = -15$$
$$(4 - 9d^2)(4 - d^2) = -15 \quad [\because \alpha = 2]$$
$$16 - 40d^2 + 9d^4 = -15$$

i.e., $$9d^4 - 40d^2 + 31 = 0$$

$$d^2 = \frac{40 \pm \sqrt{1600 - 1116}}{18} = \frac{40 \pm 22}{18} = \frac{62}{18} \text{ or } \frac{18}{18}$$

$$d^2 = \frac{31}{9} \text{ or } 1$$

$$d = \pm 1 \text{ or } \frac{\sqrt{31}}{3}$$

when $d = 1$ and $\alpha = 2$, the roots are $2 - 3, 2 - 1, 2 + 1, 2 + 3$.

i.e., $-1, 1, 3, 5$.

These are the roots of the equation (2). The roots of the given equation are the reciprocals of the roots of the equation (2).

$\therefore$ The roots of the given equation are $-1, 1, \frac{1}{3}, \frac{1}{5}$.

Example 3 *Find the condition that the roots of $x^3 + 3px^2 + 3qx + r = 0$ may be in H.P.*

Solution

Given $$x^3 + 3px^2 + 3qx + r = 0 \quad \ldots (1)$$

Change x to $\frac{1}{x}$ in (1) we get

$$\left(\frac{1}{x}\right)^3 + 3p\left(\frac{1}{x}\right)^2 + 3q\left(\frac{1}{x}\right) + r = 0$$

i.e., $$rx^3 + 3qx^2 + 3px + 1 = 0 \quad \ldots (2)$$

The roots of (2) are in A.P. [$\because$ The roots of (2) are reciprocal of (1)]

Let the roots of (2) be $\alpha - d, \alpha, \alpha + d$.

Sum of the roots $$\alpha - d + \alpha + \alpha + d = \frac{-3q}{r}$$
$$3\alpha = \frac{-3q}{r}$$
$$\alpha = \frac{-q}{r} \quad \ldots (3)$$

Since α is a root of (2), we get by substituting (3) in (2),

$$r\left(\frac{-q}{r}\right)^3 + 3q\left(\frac{-q}{r}\right)^2 + 3p\left(\frac{-q}{r}\right) + 1 = 0$$

i.e., $$2q^3 - 3pqr + r^2 = 0$$

which is the required condition.

Exercises

I. Find the equation whose roots are equal in magnitude but opposite in sign to the roots of the following equations.

1. $x^{10} - 12x^8 + 40x^4 - 15x + 20 = 0$
   [Ans. $x^{10} - 12x^8 + 40x^4 + 15x + 20 = 0$]
2. Change the signs of the roots of the equation $x^7 + 4x^5 + x^3 - 2x^2 + 7x + 3 = 0$. *[Ans. $x^7 + 4x^5 + x^3 + 2x^2 + 7x - 3 = 0$]*
3. $x^5 + 6x^4 + 6x^3 - 7x^2 + 2x - 1 = 0$
   [Ans. $x^5 - 6x^4 + 6x^3 + 7x^2 + 2x + 1 = 0$]

II. Multiply the roots of the equation.

1. $x^7 - 5x^4 + 13x^2 - 11 = 0$ by 2. *[Ans. $x^7 - 40x^4 + 416x^2 - 1408 = 0$]*
2. $x^3 - 6x^2 + 12x - 8 = 0$ by 10. *[Ans. $x^3 - 60x^2 + 1200x - 8000 = 0$]*
3. $x^5 - 4x^5 + 3x^2 - 4x + 6 = 0$ by 2.
   [Ans. $x^4 - 8x^5 + 12x^2 - 32x + 96 = 0$]
4. Remove the fractional coefficients from $x^5 - \frac{1}{3}x^4 + \frac{25}{27}x^2 + \frac{14}{81}x - \frac{8}{81} = 0$ such that the coefficient of the leading term remains unit.
   [Ans. $x^5 - x^4 + 25x^2 + 14x - 24 = 0$]
5. Remove the fractional coefficients from $x^3 - \frac{3}{2}x^2 - \frac{1}{16}x + \frac{1}{32} = 0$ such that the coefficient of the leading term remains unity.
6. Remove the fractional coefficients from $x^5 + \frac{4}{3}x^4 + \frac{2}{9}x^3 + \frac{1}{12}x^2 + \frac{1}{36} = 0$ such that the coefficient of the leading term remains unity.

8. If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$ find the value of *(i)* $\Sigma\alpha^3$ *(ii)* $\Sigma \frac{\beta^2 + \gamma^2}{\beta\gamma}$ *(iii)* $\Sigma \frac{1}{\alpha^2 \beta^2}$.

[Ans. (i) $3pq - p^3 - 3r$ *(ii)* $\frac{pq}{r} - 3$ *(iii)* $\frac{p^2 - 2q}{r^2}$ *]*

9. If α, β, γ are the roots of $x^3 + qx + r = 0$, find the value of $\frac{1}{\alpha + \beta} + \frac{1}{\beta + \gamma} + \frac{1}{\gamma + \alpha}$. *[Ans.* $\frac{q}{r}$ *]*

10. If $\alpha, \beta, \gamma, \delta$ are the roots of $x^4 - 4x^2 - x + 2 = 0$ find the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}$. *[Ans.* $\frac{17}{4}$ *]*

11. If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$ find the equation whose roots are $\alpha(\beta + \gamma), \beta(\gamma + \alpha), \gamma(\alpha + \beta)$.

[Ans. $x^3 - 2qx^2 + (q^2 + pr)x + r^2 - pqr = 0$*]*

❑ TRANSFORMATION OF EQUATIONS

Let $$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0 \quad \ldots (1)$$

›e a given equation. Let its roots be $\alpha_1, \alpha_2, \ldots, \alpha_n$. It is possible to ransform this equation into another equation whose roots are the roots of 1) with a given relation. Such transformation often helps us to solve quations easily. Here we shall explain some of the most elementary ansformations of equations.

I. To transform an equation into another equation whose roots are ıe roots of the given equation with their signs changed.

Let the given equation be

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0 \quad \ldots (1)$$

Let x be a root of the given equation and y be a root of the required uation.

Then $y = -x$ or $x = -y$. Put $x = -y$ in (1) we get,

$$a_0y^n - a_1y^{n-1} + a_2y^{n-2} - a_3y^{n-3} + \ldots = 0$$

$$a_0x^n - a_1x^{n-1} + a_2x^{n-2} - a_3x^{n-3} + \ldots = 0 \quad \text{(change } y \text{ to } x\text{)}$$

ich is the required equation whose roots are the roots of (1) with their ns changed.

Thus to transform an equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0 \quad \ldots (1)$$

another equation whose roots are the roots of (1) with their signs nged then just change the sign of the odd powers of x.

Example 1 *If the roots of $x^3 - 12x^2 + 23x + 36 = 0$ are $-1, 4, 9$ find the equation whose roots are $1, -4, -9$.*

Solution

Given $$x^3 - 12x^2 + 23x + 36 = 0 \quad \ldots (1)$$

The roots are $-1, 4, 9$.

Now we find an equation whose roots are $1, -4, -9$ *i.e.*, to find an equation whose roots are the roots of (1) but the signs are changed. Hence in (1) we have to change the sign of odd powers of x.

Hence the required equation is

$$-x^3 - 12x^2 - 23x + 36 = 0$$

i.e., $$x^3 + 12x^2 + 23x - 36 = 0$$

This gives the required equation.

Example 2 *Find the equation whose roots are $-1, -6, 2, -3$ if the roots of the equation $x^4 - 8x^3 + 7x^2 + 36x - 36 = 0$ are $1, -2, 3, 6$.*

Solution

The roots of the required equation are the negative of the roots of the given equation. Hence the required equation can be obtained by changing the signs of odd powers of x of the given equation.

Hence the required equation is

$$x^4 + 8x^3 + 7x^2 - 36x - 36 = 0$$

Example 3 *Find the equation whose roots are equal in magnitude but opposite in sign to the roots of the equation $x^{10} - 12x^8 + 40x^4 - 15x + 20 = 0$.*

Solution

To get the required equation we have to change the sign of odd powers of x in the given equation.

Hence the required equation is

$$x^{10} - 12x^8 + 40x^4 + 15x + 20 = 0$$

II. To transform an equation into another whose roots are *m* times those of the given equation.

Let $$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0 \quad \ldots (1)$$

If x is a root of (1) then $y = mx$ is a root of the required equation. Putting $x = \frac{y}{m}$ in (1) we get