Now, $\dfrac{xf^{1}(x)}{f(x)}$ can be evaluated as follows :

$$\begin{array}{r|l} & 4+5+25+122+609+\ldots.. \\ 1-5+0+1-1 & 4-15+0+1+0 \\ & \underline{4-20+0+4-4} \\ & 5+0-3+4 \\ & \underline{5-25+0+5-5} \\ & 25-3-1+5 \\ & \underline{25-125+0+25-25} \\ & 122-1-20+25 \\ & \underline{122-610+0+122-122} \\ & 609-20-97+122 \\ & \underline{609-3045+0+609-609} \\ & \ldots\ldots\ldots\ldots \end{array}$$

Therefore,

$$\frac{xf^{1}(x)}{f(x)} = 4+\frac{5}{x}+\frac{25}{x^2}+\frac{122}{x^3}+\frac{609}{x^4}+\ldots\ldots$$

Sum of the fourth powers of the roots = coefficient of x^{-4}.

= 609.

5. If $\alpha+\beta+\gamma=1$, $\alpha^2+\beta^2+\gamma^2=2$, $\alpha^3+\beta^3+\gamma^3=3$. Find $\alpha^4+\beta^4+\gamma^4$.

Solution:

Let $x^3+P_1x^2+P_2x+P_3=0$ be the equation whose roots are α,β,γ, then

$$\alpha+\beta+\gamma=-P_1 \Rightarrow P_1=-1$$

By Newton's theorem,

$$S_2+S_1P_1+2P_2=0$$

i.e., $2+1.(-1)+2P_2=0 \Rightarrow P_2 = {}^{-1}/_{2}$

Again, by Newton's theorem

$$S_3+S_2P_1+S_1P_2+3P_3=0$$

i.e., $3+2.-1+1.{}^{-1}/_{2}+3.P_3=0$

$\Rightarrow P_3 = {}^{-1}/_{6}$

Also $S_4+S_3P_1+S_2P_2+S_1P_3=0$ (By Newton's theorem for the case $r<n$)

Substituting and simplifying, we obtain $S_4 = {}^{25}/_{6}$

Thus $\alpha^4+\beta^4+\gamma^4 = \dfrac{25}{6}$

6. Calculate the sum of the cubes of the roots of $x^4+2x+3=0$

The required equation is $(x-a)(x-b)(x-c)=0$

i.e., $x^3-(a+b+c)x^2+(ab+bc+ac)x-abc=0$ .... (1)

$$a+b+c=\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5+\alpha^6=\frac{\alpha(\alpha^6-1)}{\alpha-1}=\frac{\alpha^7-\alpha}{\alpha-1}=\frac{1-\alpha}{\alpha-1}=-1$$

(Since α is a root of $x^7-1=0$, we have $\alpha^7=1$)

Similarly we can find that $ab+bc+ac=-2$, $abc=1$.

Thus from (1), the required equation is

$$x^3+x^2-2x-1=0$$

3. If α, β, γ are the roots of $x^3+3x^2+2x+1=0$, find $\Sigma\alpha^3$ and $\Sigma\overline{\alpha}^{2}$.

Solution:

Here $\alpha+\beta+\gamma=-3$, $\alpha\beta+\beta\gamma+\alpha\gamma=2$, $\alpha\beta\gamma=-1$

Using the identity $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ac)$, we find that

$$\sum\alpha^3 = (\alpha+\beta+\gamma)\ [\alpha^2+\beta^2+\gamma^2-(\alpha\beta+\beta\gamma+\alpha\gamma)]+3\alpha\beta\gamma$$

$$= (\alpha+\beta+\gamma)\left[[(\alpha+\beta+\gamma)^2-2\ (\alpha\beta+\beta\gamma+\alpha\gamma)]-(\alpha\beta+\beta\gamma+\alpha\gamma)\right]+3\alpha\beta\gamma$$

$$=-3[(9-4)-2]-3$$

$$=-9-3=-12$$

Also, $$\sum\alpha^{-2}=\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2} = \frac{\beta^2\gamma^2+\alpha^2\gamma^2+\beta^2\alpha^2}{\alpha^2\beta^2\gamma^2}$$

$$=\frac{(\alpha\beta+\beta\gamma+\alpha\gamma)^2-2\Sigma\alpha^2\beta\gamma}{\alpha^2\beta^2\gamma^2}$$ .... (1)

We have:

$$\Sigma\alpha^2\beta\gamma=(\alpha+\beta+\gamma)\alpha\beta\gamma=-3.-1=3$$

$$(1)\Rightarrow\ \Sigma\overline{\alpha}^{2}=\frac{4-2.3}{1}=-2$$

4. Find the sum of the 4th powers of the roots of the equation $x^4-5x^3+x-1=0$.

Solution:

Let $f(x)=x^4-5x^3+x-1=0$

Then $f^1(x)=4x^3-15x^2+1$

$$\Rightarrow \quad S_r + S_{r-1}P_1 + S_{r-2}P_2 + \ldots\ldots + S_1P_{r-1} + rP_r = 0$$

If $r > n$, then $r-1 > n-1$.

Equating coefficients of y^{r-1} on both sides,

$$0 = -S_r - S_{r-1}P_1 - S_{r-2}P_2 - \ldots\ldots - S_{r-n}P_n$$

$$\text{i.e.,} \quad S_r + S_{r-1}P_1 + S_{r-2}P_2 + \ldots\ldots + S_{r-n}P_n = 0$$

Remark:

To find the sum of the negative powers of the roots of $f(x) = 0$, put $x = \frac{1}{y}$ and find the sums of the corresponding positive powers of the roots of the new equation.

Illustrative Examples

1. If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, find the value of the following in terms of the coefficients.

$(i)\Sigma\frac{1}{\beta\gamma}(ii)\Sigma\frac{1}{\alpha}$ $(iii)\Sigma\alpha^2\beta$

Solution:

Here $\alpha+\beta+\gamma = -p$, $\alpha\beta+\beta\gamma+\alpha\gamma = q$, $\alpha\beta\gamma = -r$

(i) $$\Sigma\frac{1}{\beta\gamma} = \frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\alpha\gamma} = \frac{\alpha+\beta+\gamma}{\alpha\beta\gamma} = \frac{-p}{-r} = \frac{p}{r}$$

(ii) $$\Sigma\frac{1}{\alpha} = \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\alpha\beta+\beta\gamma+\alpha\gamma}{\alpha\beta\gamma} = \frac{q}{-r} = -\frac{q}{r}$$

(iii) $$\Sigma\alpha^2\beta = \alpha^2\beta+\beta^2\alpha+\gamma^2\alpha+\gamma^2\beta+\alpha^2\gamma+\beta^2\gamma$$

$$= (\alpha\beta+\beta\gamma+\alpha\gamma)(\alpha+\beta+\gamma)-3\alpha\beta\gamma = (q\,.\,-p) - 3\,(-r) = 3r - pq\ .$$

2. If α is an imaginary root of the equation $x^7 - 1 = 0$ form the equation whose roots are $\alpha+\alpha^6, \alpha^2+\alpha^5, \alpha^3+\alpha^4$.

Solution:

Let $a = \alpha+\alpha^6 \quad b = \alpha^2+\alpha^5 \quad c = \alpha^3+\alpha^4$

1.3. Symmetric Functions of the Roots

Consider the expressions like $\alpha^2+\beta^2+\gamma^2, (\beta-\gamma)^2+(\gamma-\alpha)^2+(\alpha-\beta)^2$, $(\beta+\gamma)(\gamma+\alpha)(\alpha-\beta)$. Each of these expressions is a function of α, β, γ with the property that if any two of α, β, γ are interchanged, the function remains unchanged.

Such functions are called **symmetric functions.**

Generally, a function $f(\alpha_1, \alpha_2, ..., \alpha_n)$ is said to be a symmetric function of $\alpha_1, \alpha_2, ..., \alpha_n$ if it remains unchanged by interchanging any two of $\alpha_1, \alpha_2, ..., \alpha_n$.

Remark:

The expressions $S_1, S_2, ..., S_n$ where S_r is the sum of the products of $\alpha_1, \alpha_2, ..., \alpha_n$ taken r at a time, are symmetric functions. These are called **elementary symmetric functions.**

Now we discuss some results about the sums of powers of the roots of a given polynomial equation.

1.3.1. Theorem

The sum of the r^{th} powers of the roots of the equation $f(x) = 0$ is the coefficient of x^{-r} in the expansion of $\dfrac{xf'(x)}{f(x)}$ in descending powers of x.

Proof:

Let $f(x) = 0$ be the given n^{th} degree equation and let its roots be $\alpha_1, \alpha_2, ..., \alpha_n$ then, $f(x) = a_0(x-\alpha_1)(x-\alpha_2)....(x-\alpha_n)$ where a_0 is some constant.

Taking logarithm, we obtain

$$\log f(x) = \log a_0 + \log(x-\alpha_1) + ... + \log(x-\alpha_n)$$

Differentiating w.r.t. x, we have:

$$\frac{f'(x)}{f(x)} = \frac{1}{x-\alpha_1} + ... + \frac{1}{x-\alpha_n}$$

Multiplying by x,

$$\frac{xf'(x)}{f(x)} = \frac{x}{x-\alpha_1} + ... + \frac{x}{x-\alpha_n}$$

$$= \left(1-\frac{\alpha_1}{x}\right)^{-1} + ... + \left(1-\frac{\alpha_n}{x}\right)^{-1}$$

$$= \left(1+\frac{\alpha_1}{x}+\frac{\alpha_1^2}{x^2}+ ....\right) + ... + \left(1+\frac{\alpha_n}{x}+\frac{\alpha_n^2}{x^2}+ ....\right)$$

5. Solve the equation $x^3 - 9x^2 + 14x + 24 = 0$, given that two of whose roots are in the ratio 3: 2.

Solution:

Let the roots be $3\alpha, 2\alpha, \beta$

Then, $3\alpha + 2\alpha + \beta = 5\alpha + \beta = 9$ .... (1)

$$3\alpha.2\alpha + 2\alpha.\beta + 3\alpha.\beta = 14$$

i.e., $6\alpha^2 + 5\alpha\beta = 14$ .... (2)

and $3\alpha.2\alpha.\beta = 6\alpha^2\beta = -24$

$\Rightarrow \alpha^2\beta = -4$ .... (3)

From (1), $\beta = 9 - 5\alpha$. Substituting this in (2), we obtain

$$6\alpha^2 + 5\alpha(9 - 5\alpha) = 14$$

i.e., $19\alpha^2 - 45\alpha + 14 = 0$. On solving we get $\alpha = 2 \ or \ \frac{7}{19}$.

When $\alpha = \frac{7}{19}$, from (1), we get $\beta = \frac{136}{19}$. But these values do not satisfy (3).

So, $\alpha = 2$, then from (1), we get $\beta = -1$

Therefore, the roots are 4, 6, -1.

1. If the roots of the equation $x^3 + px^2 + qx + r = 0$ are in arithmetic progression, show that $2p^3 - 9pq + 27r = 0$.

Solution:

Let the roots of the given equation be $a - d,\ a,\ a + d$.

Then $S_1 = a - d + a + a + d = 3a = -p \Rightarrow a = \frac{-p}{3}$

Since a is a root, it satisfies the given polynomial

$$\Rightarrow \left(-p/3\right)^3 + p.\left(-p/3\right)^2 + q.\left(-p/3\right) + r = 0$$

On simplification, we obtain $2p^3 - 9pq + 27r = 0$.

2. Solve $27x^3 + 42x^2 - 28x - 8 = 0$, given that its roots are in geometric progression.

Solution:

Let the roots be $\frac{a}{r}, a, ar$

Then , $\frac{a}{r}.a.ar = a^3 = \frac{8}{27} \Rightarrow a = \frac{2}{3}$

Since $a = \frac{2}{3}$ is a root, $\left(x - \frac{2}{3}\right)$ is a factor. On division, the other factor of the polynomial is $27x^2 + 60x + 12$.

Its roots are $\frac{-60 \pm \sqrt{60^2 - 4\times 27\times 12}}{2\times 27} = \frac{-2}{9} \quad or \quad -2$

Hence the roots of the given polynomial eqution are $\frac{-2}{9},\ -2,\ \frac{2}{3}$.

Solution:

Let the given equation be

$$x^4 + P_1x^3 + P_2x^2 + P_3x + P_4 = 0$$

Here $P_1 = P_2 = 0$, $P_3 = 2$ and $P_4 = 3$

By Newton's theorem, $S_3 + S_2P_1 + S_1P_2 + 3P_3 = 0$

$$\text{i.e.,}\quad S_3 + 0 + 0 + 3 \cdot 2 = 0$$

$$\Rightarrow S_3 = -6$$

i.e., sum of the cubes of the roots of $x^4 + 2x + 3 = 0$, is -6.

Now, $\dfrac{xf^{1}(x)}{f(x)}$ can be evaluated as follows :

$$
\begin{array}{r l}
 & \underline{4+5+25+122}+609+\ldots.. \\
1-5+0+1-1 \big) & 4-15+0+1+0 \\
 & \underline{4-20+0+4-4} \\
 & \quad 5+0-3+4 \\
 & \quad \underline{5-25+0+5-5} \\
 & \qquad 25-3-1+5 \\
 & \qquad \underline{25-125+0+25-25} \\
 & \qquad\quad 122-1-20+25 \\
 & \qquad\quad \underline{122-610+0+122-122} \\
 & \qquad\qquad 609-20-97+122 \\
 & \qquad\qquad \underline{609-3045+0+609-609} \\
 & \qquad\qquad\qquad \ldots\ldots\ldots\ldots
\end{array}
$$

Therefore,

$$\frac{xf^{1}(x)}{f(x)} = 4+\frac{5}{x}+\frac{25}{x^2}+\frac{122}{x^3}+\frac{609}{x^4}+\ldots\ldots$$

Sum of the fourth powers of the roots = coefficient of x^{-4}.

$$= 609.$$

5. If $\alpha+\beta+\gamma=1$, $\alpha^2+\beta^2+\gamma^2=2$, $\alpha^3+\beta^3+\gamma^3=3$. Find $\alpha^4+\beta^4+\gamma^4$.

Solution:

Let $x^3 + P_1x^2 + P_2x + P_3 = 0$ be the equation whose roots are α, β, γ, then

$$\alpha+\beta+\gamma = -P_1 \Rightarrow P_1 = -1$$

By Newton's theorem,

$$S_2 + S_1P_1 + 2P_2 = 0$$

$$\text{i.e.,} \quad 2 + 1.(-1) + 2\,P_2 = 0 \Rightarrow P_2 = {}^{-1}/_{2}$$

Again, by Newton's theorem

$$S_3 + S_2P_1 + S_1P_2 + 3P_3 = 0$$

$$\text{i.e.,} \quad 3 + 2.-1 + 1.{}^{-1}/_{2} + 3.P_3 = 0$$

$$\Rightarrow P_3 = {}^{-1}/_{6}$$

Also $S_4 + S_3P_1 + S_2P_2 + S_1P_3 = 0$ (By Newton's theorem for the case $r < n$)

Substituting and simplifying, we obtain $S_4 = {}^{25}/_{6}$

Thus $\alpha^4+\beta^4+\gamma^4 = \dfrac{25}{6}$

6. Calculate the sum of the cubes of the roots of $x^4 + 2x + 3 = 0$

1.3.2. Theorem (Newton's Theorem on the Sum of the Powers of the Roots)

If $\alpha_1, \alpha_2, ..., \alpha_n$ are the roots of the equation $x^n + P_1 x^{n-1} + P_2 x^{n-2} + ... + P_n = 0$,

and $S_r = \alpha_1^r + ... + \alpha_n^r$. Then, $S_r + S_{r-1}P_1 + ... + S_1 P_{r-1} + rP_r = 0$, *if* $r \leq n$.

and $S_r + S_{r-1}P_1 + S_{r-2}P_2 + ... + S_{r-n}P_n = 0$ if $r > n$.

Proof:

We have $x^n + P_1 x^{n-1} + P_2 x^{n-2} + ... + P_n = (x-\alpha_1)(x-\alpha_2).....(x-\alpha_n)$

Put $x = \dfrac{1}{y}$

$$\Rightarrow \quad \frac{1}{y^n} + \frac{P_1}{y^{n-1}} + \frac{P_2}{y^{n-2}} + ... + P_n = \left(\frac{1}{y} - \alpha_1\right)\left(\frac{1}{y} - \alpha_2\right).....\left(\frac{1}{y} - \alpha_n\right),$$

and then multiplying by y^n, we obtain:

$$1 + P_1 y + P_2 y^2 + ... + P_n y^n = (1-\alpha_1 y)(1-\alpha_2 y).....(1-\alpha_n y)$$

Taking logarithm and differentiating w.r.t y, we get

$$\frac{P_1 + 2P_2 y + 3P_3 y^2 + ... + nP_n y^{n-1}}{1 + P_1 y + P_2 y^2 + ... + P_n y^n} = \frac{-\alpha_1}{1-\alpha_1 y} + \frac{-\alpha_2}{1-\alpha_2 y} + ... + \frac{-\alpha_n}{1-\alpha_n y}$$

$$= -\alpha_1(1-\alpha_1 y)^{-1} - \alpha_2(1-\alpha_2 y)^{-1} - ...... - \alpha_n(1-\alpha_n y)^{-1}$$

$$= -\alpha_1(1 + \alpha_1 y + \alpha^2 y^2 + ...) - \alpha_2(1 + \alpha_2 y + \alpha_2^2 y^2 + ...) - \\ ...... - \alpha_n(1 + \alpha_n y + \alpha_n^2 y^2 + ...)$$

$$= -S_1 - S_2 y - S_3 y^2 - ..... - S_{r+1} y^r - ...$$

Cross – multiplying, we get

$$P_1 + 2P_2 y + 3P_3 y^2 + ... + nP_n y^{n-1} = -(1 + P_1 y + P_2 y^2 + ... + P_n y^n)$$

$$[S_1 + S_2 y + ... + S_{r+1} y^r + ...]$$

Equating coefficients of like powers of y, we see that

$$P_1 = -S_1 \quad \Rightarrow \quad S_1 + 1.P_1 = 0$$

$$2P_2 = -S_2 - S_1 P_1 \quad \Rightarrow \quad S_2 + S_1 P_1 + 2P_2 = 0$$

$$3p_3 = -S_3 - S_2 P_1 - S_1 P_2 \quad \Rightarrow \quad S_3 + S_2 P_1 + S_1 P_2 + 3P_3 = 0, \text{ and so on}.$$

If $r \leq n$, equating coefficients of y^{r-1} on both sides,

$$rP_r = -S_r - S_{r-1}P_1 - S_{r-2}P_2 - ...... - S_1 P_{r-1}$$

3. Solve the equation $15x^3 - 23x^2 + 9x - 1 = 0$ whose roots are in harmonic progression.

Solution:

[Recall that if a, b, c are in harmonic progression, then $1/a$, $1/b$, $1/c$ are in arithmetic progression and hence $b = \frac{2ac}{a+c}$]

Let α, β, γ be the roots of the given polynomial.

Then $\alpha\beta + \beta\gamma + \alpha\gamma = \frac{9}{15}$ .... (1)

$$\alpha\beta\gamma = \frac{1}{15} \quad \text{.......... (2)}$$

Since α, β, γ are in harmonic progression, $\beta = \frac{2\alpha\gamma}{\alpha + \gamma}$

$$\Rightarrow \alpha\beta + \beta\gamma = 2\alpha\gamma$$

Substitute in (1), $2\alpha\gamma + \alpha\gamma = \frac{.9}{15} \Rightarrow 3\alpha\gamma = \frac{9}{15}$

$$\Rightarrow \alpha\gamma = \frac{3}{15}.$$

Substitute in (2), we obtain $\frac{3}{15}\beta = \frac{1}{15}$

$\Rightarrow \beta = \frac{1}{3}$ is a root of the given polynomial.

Proceeding as in the above problem, we find that the roots are $\frac{1}{3}$, 1, $\frac{1}{5}$.

4. Show that the roots of the equation $ax^3 + bx^2 + cx + d = 0$ are in geometric progression, then $c^3a = b^3d$.

Solution:

Suppose the roots are $\frac{k}{r},\ k,\ kr$

Then $\frac{k}{r}.k.kr = \frac{-d}{a}$

i.e., $k^3 = \frac{-d}{a}$

Since k is a root, it satisfies the polynomial equation,

$$ak^3 + bk^2 + ck + d = 0$$

$$a\left(\frac{-d}{a}\right) + bk^2 + ck + d = 0$$

$$\Rightarrow bk^2 + ck = 0 \quad \Rightarrow bk^2 = -ck$$

$$\Rightarrow (bk^2)^3 = (-ck)^3 \quad i.e., \ b^3k^6 = -c^3k^3$$

$$\Rightarrow b^3\frac{d^2}{a^2} = -c^3\left(\frac{-d}{a}\right)$$

$$\Rightarrow \quad \frac{b^3d}{a} = c^3 \ \Rightarrow \ b^3d = c^3a.$$

1.2. Relation between the Roots and Coefficients of a Polynomial Equation

Consider the polynomial function $f(x) = a_o x^n + a_1 x^{n-1} + \ldots + a_n$, $a_0 \neq 0$

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of $f(x) = 0$.

Then we can write $f(x) = a_o(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$

Equating the two expressions for $f(x)$, we obtain:

$$a_o x^n + a_1 x^{n-1} + \ldots + a_n = a_o(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$$

Dividing both sides by a_0,

$$x^n + \left(\frac{a_1}{a_o}\right)x^{n-1} + \ldots + \left(\frac{a_n}{a_o}\right) = (x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$$

$$= x^n - S_1 x^{n-1} + S_2 x^{n-2} - \ldots + (-1)^n S_n$$

where S_r stands for the sum of the products of the roots $\alpha_1, \ldots, \alpha_n$ taken r at a time.

Comparing the coefficients on both sides, we see that

$$S_1 = \frac{-a_1}{a_o}, \quad S_2 = \frac{a_2}{a_o}, \ldots \quad S_n = (-1)^n \frac{a_n}{a_o}.$$