UNIT - V

Logarithm of complex numbers

Def: let $z = r(\cos\theta + i\sin\theta)$ be a non-zero complex number we define $\log z = \log r + i\theta$

$z = x + iy$

$x = r\cos\theta$; $y = r\sin\theta$

$z = r(\cos\theta + i\sin\theta)$

$z = r \cdot e^{i\theta}$. $\qquad e^{i\theta} = \cos\theta + i\sin\theta$

Taking log on both sides

$\log z = \log(re^{i\theta})$

$= \log r + \log e^{i\theta}$

$$\boxed{\log z = \log r + i\theta}$$

NOTE:

If we denote general value of $\log z$ by $\log z$.

i) then $\log z = \log z + i2n\pi$, $n \in z$

$\log z = \log r + i\theta + i2n\pi$

$\log z = \log r + i(\theta + 2n\pi)$.

ii) $z = x + iy$

$x + iy = r(\cos\theta + i\sin\theta)$

$|x + iy| = |r|\,|\cos\theta + i\sin\theta|$

$\sqrt{x^2+y^2} = |r|\sqrt{1}$

$\sqrt{x^2+y^2} = r$

$\therefore \theta = \tan^{-1}(y/x)$

$\log z = \log\sqrt{x^2+y^2} + i\left[\tan^{-1}\left(\frac{y}{x}\right) + 2n\pi\right]$

Theorem:

$\log(z_1 z_2) = \log z_1 + \log z_2$

Proof:

$z = r(\cos\theta + i\sin\theta)$

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2 (\cos\theta_1 \cos\theta_2 + i\cos\theta_1 \sin\theta_2 + i\sin\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2)$

$= r_1 r_2 [(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2)]$

$= r_1 r_2 [(\cos(\theta_1 + \theta_2)) + i\sin(\theta_1 + \theta_2)]$

$z_1 z_2 = r_1 r_2 [e^{i(\theta_1 + \theta_2)}]$

Taking log on both sides.

$\log(z_1 z_2) = \log(r_1 r_2) \cdot \log[e^{i(\theta_1 + \theta_2)}]$

$\log(z_1 z_2) = \log r_1 + \log r_2 + i(\theta_1 + \theta_2)$

$= \log r_1 + \log r_2 + i\theta_1 + i\theta_2$

$= \log r_1 + \log r_2 + i(\theta_1 + 2n\pi) + i(\theta_2 + 2n\pi)$

$= [\log r_1 + i(\theta_1 + 2n\pi)] + [\log r_2 + i(\theta_2 + 2n\pi)]$

$\log(z_1 z_2) = \log z_1 + \log z_2$

1. Find $\log(1+i)$

Soln:

W.K.T: $\log z = \log r + i(\theta + 2n\pi)$.

$r = \sqrt{x^2+y^2}$

$\theta = \tan^{-1}(y/x)$

$z = 1+i$.

$x = 1,\ y = 1$

$r = \sqrt{1+1}$

$\boxed{r = \sqrt{2}}$

$\theta = \tan^{-1}(1/1)$

$\boxed{\theta = \pi/4}$

$\log z = \log(\sqrt{2}) + i\left(\frac{\pi}{4} + 2n\pi\right)$

$= \log\sqrt{2} + i\left(\frac{\pi + 8n\pi}{4}\right)$

$= \log\sqrt{2} + \frac{\pi i}{4}(1+8n)$.

2. Find $\log(i)$

Soln:

W.K.T: $\log z = \log r + i(\theta + 2n\pi)$

$x = 0,\ y = 1$

$r = \sqrt{x^2+y^2}$

$\boxed{r = \sqrt{1}}$ $\boxed{r = 1}$

$\theta = \tan^{-1}(y/x)$

$= \tan^{-1}(1/0)$

$= \tan^{-1}(\infty)$

$$\boxed{\theta = \pi/2}$$

$$\log z = \log 1 + i\left(\pi/2 + 2n\pi\right)$$

$$= 0 + i\left(\frac{\pi + 4n\pi}{2}\right)$$

$$\log z = \frac{\pi i}{2}(1+4n).$$

3. P.T:

i) $\log 1 = i2n\pi$

<u>Soln:</u>

N.K.T: $\log z = \log r + i(\theta + 2n\pi)$

$x = 1, \quad y = 0$

$r = \sqrt{x^2+y^2}$

$r = \sqrt{1}$

$\boxed{r = 1}$

$\theta = \tan^{-1}(0/1)$

$\theta = \tan^{-1}(0)$

$\boxed{\theta = 0}$

$\log z = \log 1 + i(0 + 2n\pi)$

$= 0 + i2n\pi$

$\log z = i2n\pi$

ii) $\log -1 = i(2n+1)\pi$

<u>Soln:</u>

N.K.T: $\log z = \log r + i(\theta + 2n\pi)$

$x = -1, \quad y = 0$

$r = \sqrt{x^2+y^2}$

$r = \sqrt{1}$

$\boxed{r = \pm 1}$

$\theta = \tan^{-1}(y/x)$

$= \tan^{-1}(0/-1)$

$= \tan^{-1}(0)$

$\boxed{\theta = \pi}$

if $\tan^{-1}(-1)$ the value of $\theta = \pi$

$\log z = \log 1 + i(\pi + 2n\pi)$

$= 0 + i(2n+1)\pi$

$\log z = i(2n+1)\pi$

iii) $\log(-1+i) = \frac{1}{2}\log 2 + i(2n\pi + 3\pi/4)$

Soln:

W.K.T: $\log z = \log r + i(\theta + 2n\pi)$.

$x = -1,\ y = 1$

$r = \sqrt{1+1}$

$\boxed{r = \sqrt{2}}$

$\theta = \tan^{-1}(1/-1)$

$\theta = \tan^{-1}(-1)$

$\boxed{\theta = 3\pi/4}$

$\log z = \log(2)^{1/2} + i\left(\frac{3\pi}{4} + 2n\pi\right)$

$= \frac{1}{2}\log 2 + i\cancel{\left(\frac{3\pi + 8n\pi}{4}\right)} = \frac{1}{2}\log 2 + i\left(2n\pi + \frac{3\pi}{4}\right)$

v) $\log\sqrt{i} = i(4n+1)\frac{\pi}{2}$

vi) $\log(-1-i) = \frac{1}{2}\log 2 + i(2n\pi - 3\pi/4)$

iv) $\log(-i) = i(4n+1)\pi/2$

Soln:

W.K.T: $\log z = \log r + i(\theta + 2n\pi)$

$x = 0,\ y = -1$

$r = \sqrt{x^2+y^2}$

$r = \sqrt{1}$

$\boxed{r = 1}$

$\theta = \tan^{-1}(y/x)$

$= \tan^{-1}(-1/0) = \tan^{-1}(\infty)$

$\boxed{\theta = \pi/2}$

$$\log z = \log 1 + i\left(\frac{\pi}{2} + 2n\pi\right)$$
$$= 0 + i\left(\frac{\pi + 4n\pi}{2}\right)$$
$$= i(4n+1)\pi/2$$

v) $\log \sqrt{i} = i(4n+1)\pi/2$

$x = 0,\ y = \sqrt{1} = 1$

$r = \sqrt{0+1}$

$\boxed{r = 1}$

$\theta = \tan^{-1}(1/0)$

$\theta = \tan^{-1}(\infty)$

$\boxed{\theta = \pi/2}$

$$\log z = \log 1 + i\left(\frac{\pi}{2} + 2n\pi\right)$$
$$= 0 + i\left(\frac{\pi + 4n\pi}{2}\right)$$
$$= i(4n+1)\pi/2$$

vi) $\log(-1-i) = \frac{1}{2}\log 2 + i(2n\pi - 3\pi/4)$

$x = -1,\ y = -1$

$r = \sqrt{x^2+y^2}$

$r = \sqrt{1+1}$

$\boxed{r = \sqrt{2}}$

$\theta = \tan^{-1}(-1/_{-1})$

$\theta = \tan^{-1}(1)$ $\boxed{\theta = -3\pi/4}$

$$\log z = \log(2)^{1/2} + i(2n\pi - 3\pi/4)$$
$$= \frac{1}{2}\log 2 + (2n\pi - 3\pi/4)$$

Def: If z & ω are any two complex numbers then we define,

$$z^{\omega} = e^{\omega \log z}.$$

Note: Since $\log z$ has infinite number of values for z, z^{ω} also has infinite number of values.

1. a) If $i^{a+ib} = a+ib$. P.T $a^2+b^2 = e^{-(4n+1)\pi b}$

(or)

b) If $i^{x+iy} = A+iB$ P.T $A^2+B^2 = e^{-(4n+1)\pi y}$.

a) Soln:

Gn, $i^{a+ib} = a+ib$

$z = i$, $\omega = a+ib$

W.K.T: $z^{\omega} = e^{\omega \log z}$

$i^{a+ib} = e^{(a+ib)\log i} \rightarrow (1)$

$z = i$;

$x = 0$, $y = 1$

$r = \sqrt{x^2+y^2}$

$r = \sqrt{1}$

$\boxed{r = 1}$

$\theta = \tan^{-1}(y/x)$

$= \tan^{-1}(1/0)$

$= \tan^{-1}(\infty)$

$\boxed{\theta = \pi/2}$

$\log z = \log r + i(\theta + 2n\pi)$

$\log i = \log 1 + i(\pi/2 + 2n\pi)$

$\log i = i\left(\frac{\pi + 4n\pi}{2}\right)$

$\log i = i\pi/2\,(4n+1) \rightarrow (2)$

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$$i^{a+ib} = e^{(a+ib) i\pi/2 (4n+1)}$$

$$i^{a+ib} = e^{(ia-b)\pi/2 (4n+1)}$$

$$= e^{ia\frac{\pi}{2}(4n+1)} \cdot e^{-b\pi/2 (4n+1)}$$

$$i^{a+ib} = e^{i\theta} \cdot e^{-b\pi/2 (4n+1)}$$

where $\theta = a\pi/2 (4n+1)$

$$i^{a+ib} = (\cos\theta + i\sin\theta) e^{-b\pi/2 (4n+1)}$$

$$= \cos\theta e^{-b\pi/2 (4n+1)} + \sin\theta e^{-b\pi/2 (4n+1)}$$

$$a+ib = \cos\theta e^{-b\pi/2 (4n+1)} + \sin\theta e^{-b\pi/2 (4n+1)}$$

$$a = \cos\theta e^{-b\pi/2 (4n+1)}$$

$$b = \sin\theta e^{-b\pi/2 (4n+1)}$$

$$a^2+b^2 = \cos^2\theta e^{-2b\pi/2 (4n+1)} + \sin^2\theta e^{-2b\pi/2 (4n+1)}$$

$$= e^{-2b\pi/2 (4n+1)} \cdot (\cos^2\theta + \sin^2\theta)$$

$$a^2+b^2 = e^{-(4n+1)\pi b}.$$

2. P.T: $\log\left[\frac{(1+i)(2+i)}{3+i}\right] = i\tan^{-1}\left(\frac{4}{3}\right)$

Soln:

$$z = \frac{(1+i)(2+i)}{3+i}$$

$$= \frac{2+i+2i-1}{3+i}$$

$$= \frac{3i+1}{3+i}$$

$$= \frac{3i+1}{3+i} \times \frac{3-i}{3-i}$$

$$= \frac{9i+3+3-i}{(3)^2-(i)^2}$$

$$= \frac{8i+6}{9+1}$$

$$= \frac{2(4i+3)}{9+1}$$

$$z = \frac{4}{5}i + \frac{3}{5}$$

$$x = 3/5 \; ; \; y = 4/5$$

$$r = \sqrt{x^2+y^2}$$

$$= \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2}$$

$$= \sqrt{\left(\frac{9}{25} + \frac{16}{25}\right)}$$

$$r = \sqrt{1}$$

$$\boxed{r = 1}$$

$$\theta = \tan^{-1}(y/x)$$

$$\theta = \tan^{-1}\left(\frac{4}{5} \times \frac{5}{3}\right)$$

$$\boxed{\theta = \tan^{-1}(4/3)}$$

W.K.T: $\log z = \log r + i(\theta + 2n\pi)$ (or) $\log z = \log + i\theta$

$$\therefore \log z = \log 1 + i \tan^{-1}(4/3)$$

$$\log\left[\frac{(1+i)(2+i)}{3+i}\right] = i \tan^{-1}(4/3)$$

3. P.T: $\log z^{\omega} = \omega \log z$

Soln:

z and ω are any two complex number

$\therefore z^{\omega} = e^{\omega \log z}$

Taking log on both sides.

$\log z^{\omega} = \omega \log z.$

Summation of series:

Some of the important methods of summation are

i) Difference method

ii) Angles in arithmetics progression method.

iii) Gregory method.

Difference method:

In this method we use various trigonometric formula of split up each term unit difference of two terms such that one term of each term appears as one of the native terms of the next term and summation gives the required results.

Problem:

1. Sum the series:

$\frac{1}{\sin\theta \sin 2\theta} + \frac{1}{\sin 2\theta \sin 3\theta} + \dots + n$ terms

(or)

$cosec\,\theta\, cosec\,2\theta + cosec\,2\theta\, cosec\,3\theta + \dots + n$ terms

Soln:

let $S_n = \frac{1}{\sin\theta \sin 2\theta} + \frac{1}{\sin 2\theta \sin 3\theta} + \dots + n$ terms

Here the n^{th} terms $T_n = \frac{1}{\sin n\theta \sin(n+1)\theta}$

now $\times$ & $\div$ by $\sin\theta$

$T_n = \frac{1}{\sin\theta}\left[\frac{\sin\theta}{\sin\theta\ \sin(n+1)\theta}\right]$

now adding & sub $(n\theta)$

$T_n = \frac{1}{\sin\theta}\left[\frac{\sin\theta(\theta + n\theta - n\theta)}{\sin\theta\ \sin(n+1)\theta}\right]$

$= \frac{1}{\sin\theta}\left[\frac{\sin[(n+1)\theta - n\theta]}{\sin\theta \sin(n+1)\theta}\right]$

$= \frac{1}{\sin\theta}\left[\frac{\sin(n+1)\theta \cos n\theta - \cos(n+1)\theta \sin n\theta}{\sin n\theta \sin(n+1)\theta}\right]$

$= \frac{1}{\sin\theta}\left[\frac{\sin(n+1)\theta \cos n\theta}{\sin n\theta \sin(n+1)\theta} - \frac{\cos(n+1)\theta . \sin n\theta}{\sin n\theta \sin(n+1)\theta}\right]$

$T_n = \frac{1}{\sin\theta}\left[\cot\theta - \cot(n+1)\theta\right]$

$n=2,$

$T_2 = cosec\theta \cdot [\cot 2\theta - \cot 3\theta]$

$n=3,$

$T_3 = cosec\cdot\theta[\cot 3\theta - \cot 4\theta]$

$n=n-1,$

$T_{n-1} = cosec\theta\,[\cot(n-1)\theta - \cot n\theta]$

$T_n = cosec\,\theta\,[\cot n\theta - \cot(n+1)\theta]$

$S_n = T_1 + T_2 + T_3 + \dots + T_n$

$S_n = cosec\,\theta\,[\cot\theta - \cot(n+1)\theta]$

2. Sum of the series

$\sec\theta \sec 2\theta + \sec 2\theta \sec 3\theta + \dots + n$ terms

Soln:

let,

$S_n = \sec\theta \sec 2\theta + \sec 2\theta \sec 3\theta + \dots + n$ terms

(or)

$$S_n = \frac{1}{\cos\theta \cos 2\theta} + \frac{1}{\cos 2\theta \cos 3\theta} + \dots\dots$$

Here, the n^{th} terms.

$$T_n = \frac{1}{\cos n\theta \cos(n+1)\theta}$$

Now,

$\times$ & $\div$ by $\sin\theta$

$$T_n = \frac{1}{\sin\theta}\left[\frac{\sin\theta}{\cos n\theta \cos(n+1)\theta}\right]$$

Now adding & sub $n\theta$

$$= \frac{1}{\sin\theta}\left[\frac{\sin(\theta + n\theta - n\theta)}{\cos n\theta \cos(n+1)\theta}\right]$$

$$= \frac{1}{\sin\theta}\left[\frac{\sin[(n+1)\theta - n\theta]}{\cos n\theta \cos(n+1)\theta}\right]$$

$$= \frac{1}{\sin\theta}\left[\frac{\sin(n+1)\theta \cos n\theta - \sin n\theta \cos(n+1)\theta}{\cos n\theta \cos(n+1)\theta}\right]$$

$$\cancel{= \frac{1}{\sin\theta}\left[\frac{\sin(n+1)\theta \cos n\theta - \cos(n+1)\theta \sin n\theta}{1}\right]}$$

$$= \frac{1}{\sin\theta}\left[\frac{\sin(n+1)\theta \cos n\theta}{\cos n\theta \cos(n+1)\theta} - \frac{\cos(n+1)\theta \sin n\theta}{\cos n\theta \cos(n+1)\theta}\right]$$

$$= \frac{1}{\sin\theta}\left[\frac{\sin(n+1)\theta}{\cos(n+1)\theta} - \frac{\sin n\theta}{\cos n\theta}\right]$$

$$T_n = \operatorname{cosec}\theta\,[\tan(n+1)\theta - \tan n\theta]$$

put

$n=1$,

$$T_1 = \operatorname{cosec}\theta\,(\tan 2\theta - \tan\theta)$$

$n=2$,

$$T_2 = \operatorname{cosec}\theta\,(\tan 3\theta - \tan 2\theta)$$

$n=3$,

$$T_3 = \operatorname{cosec}\theta\,(\tan 4\theta - \tan 3\theta)$$

$\vdots$

$$T_n = \operatorname{cosec}\theta\,[\tan(n+1)\theta - \tan n\theta]$$

$$S_n = T_1 + T_2 + \cdots$$

$$S_n = \operatorname{cosec}\theta\,[\tan(n+1)\theta - \tan\theta].$$

3. $$S_n = \frac{1}{\cos\theta + \cos 3\theta} + \frac{1}{\cos\theta + \cos 5\theta} + \frac{1}{\cos\theta + \cos 7\theta} + \ldots n \text{ terms}$$

Soln:

Here $T_n = \dfrac{1}{\cos\theta + \cos(2n+1)\theta}$

N.K.T: $\cos(A+B) + \cos(A-B) = 2\cos A\cos B$

sub $A = (n+1)\theta$, $B = n\theta$

$$\cos(n+1+n)\theta + \cos(n+1-n)\theta = 2\cos(n+1)\theta\cos n\theta$$

$$\cos(2n+1)\theta + \cos\theta = 2\cos(n+1)\theta\cos n\theta.$$

$$\therefore T_n = \frac{1}{2\cos(n+1)\theta\cos n\theta}$$

Now $\times$ & $\div$ by $\sin\theta$

$$T_n = \frac{1}{2\sin\theta}\left[\frac{\sin\theta}{\cos(n+1)\theta\cos n\theta}\right]$$

add & sub $n\theta$

$$= 2\operatorname{cosec}\theta\left[\frac{\sin(\theta+n\theta-n\theta)}{\cos(n+1)\theta\cos n\theta}\right]$$

$$= 2\operatorname{cosec}\theta\left[\frac{\sin[(n+1)\theta-n\theta]}{\cos(n+1)\theta\cos n\theta}\right]$$

$$= 2\operatorname{cosec}\theta\left[\frac{\sin(n+1)\theta\cos n\theta-\cos(n+1)\theta\sin n\theta}{\cos(n+1)\theta\cos n\theta}\right]$$

$$= 2\operatorname{cosec}\theta\left[\frac{\sin(n+1)\theta\cos n\theta}{\cos(n+1)\theta\cos n\theta}-\frac{\cos(n+1)\theta\sin n\theta}{\cos(n+1)\theta\cos n\theta}\right]$$

$$= 2\operatorname{cosec}\theta\left[\frac{\sin(n+1)\theta}{\cos(n+1)\theta}-\frac{\sin n\theta}{\cos n\theta}\right]$$

$$T_n = 2\operatorname{cosec}\theta\left[\tan(n+1)\theta-\tan n\theta\right]$$

Put,

$n=1$,

$$T_1 = 2\operatorname{cosec}\theta\left[\tan 2\theta-\tan\theta\right]$$

$n=2$,

$$T_2 = 2\operatorname{cosec}\theta(\tan 3\theta-\tan 2\theta).$$

$n=3$,

$$T_3 = 2\operatorname{cosec}\theta(\tan 4\theta-\tan 3\theta)..$$

$\vdots$

$$T_n = 2\operatorname{cosec}\theta\left[\tan(n+1)\theta-\tan n\theta\right]$$

$$S_n = T_1+T_2+\cdots$$

$$S_n = 2\operatorname{cosec}\theta\left[\tan(n+1)\theta-\tan\theta\right]$$

4. Find $S_n = \sin\theta \sec 3\theta + \sin 3\theta \sec 9\theta + \sin 9\theta \sec 27\theta + \dots n$ terms.

Soln:

$$S_n = \frac{\sin\theta}{\cos 3\theta} + \frac{\sin 3\theta}{\cos 9\theta} + \frac{\sin 9\theta}{\cos 27\theta} + \dots$$

$$T_n = \frac{\sin(3^{n-1})\theta}{\cos 3^n \theta}$$

$\times$ & $\div$ $2\cos(3^{n-1}\theta)$.

$$T_n = \frac{2\sin(3^{n-1})\theta \cos(3^{n-1}\theta)}{2\cos(3^n\theta)\cos(3^{n-1}\theta)}.$$

$$= \frac{1}{2}\left[\frac{\sin(2\cdot 3^{n-1}\theta)}{\cos(3^n\theta)\cos(3^{n-1}\theta)}\right]$$

$$= \frac{1}{2}\left[\frac{\sin[(3^n\theta) - (3^{n-1}\theta)]}{\cos(3^n\theta)\cos(3^{n-1}\theta)}\right]$$

$$= \frac{1}{2}\left[\frac{\sin(3^n\theta)\cos(3^{n-1}\theta) - \cos(3^n\theta)\sin(3^{n-1}\theta)}{\cos(3^n\theta)\cos(3^{n-1}\theta)}\right]$$

$$T_n = \frac{1}{2}\left[\tan(3^n\theta) - \tan(3^{n-1}\theta)\right]$$

$n=1$,

$$T_1 = \frac{1}{2}\left[\tan 3\theta - \tan\theta\right]$$

$n=2$,

$$T_2 = \frac{1}{2}\left[\tan 9\theta - \tan 3\theta\right]$$

$n=3$,

$$T_3 = \frac{1}{2}\left[\tan 27\theta - \tan 9\theta\right]$$

$n = n-1$,

$$T_{n-1} = \frac{1}{2}\left[\tan(3^{n-1}\theta) - \tan(3^{n-2}\theta)\right]$$

$$T_n = \frac{1}{2}\left[\tan(3^n\theta) - \tan(3^{n-1}\theta)\right]$$

$$S_n = \frac{1}{2}\left[\tan(3^n\theta) - \tan\theta\right].$$

6. $S_n = \frac{1}{\sin x\cos 2x} - \frac{1}{\cos 2x\sin 3x} + \frac{1}{\sin 3x\cos 4x} - \dots n\text{ terms}$

soln:

put $x = \frac{\pi}{2} - \theta$

$$S_n = \frac{1}{\sin(\frac{\pi}{2}-\theta)\cos 2(\frac{\pi}{2}-\theta)} - \frac{1}{\cos 2(\frac{\pi}{2}-\theta)\sin 3(\frac{\pi}{2}-\theta)} + \frac{1}{\sin 3(\frac{\pi}{2}-\theta)\cos 4(\frac{\pi}{2}-\theta)} + \dots$$

$$= \frac{-1}{\cos\theta\cos 2\theta} - \frac{1}{\cos 2\theta\cos 3\theta} - \frac{1}{\cos 3\theta\cos 4\theta} - \dots$$

$$= -\left[\frac{1}{\cos\theta\cos 2\theta} + \frac{1}{\cos 2\theta\cos 3\theta} + \frac{1}{\cos 3\theta\cos 4\theta} + \dots\right]$$

$$T_n = -\frac{1}{\cos n\theta\cos(n+1)\theta}$$

$\times$ & $\div$ by $\sin\theta$

$$= \frac{-1}{\sin\theta}\left[\frac{\sin\theta}{\cos n\theta\cos(n+1)\theta}\right]$$

Add & sub by $n\theta$

$$= -\frac{1}{\sin\theta}\left[\frac{\sin(\theta + n\theta - n\theta)}{\cos n\theta\cos(n+1)\theta}\right]$$

$$= \frac{-1}{\sin\theta}\left[\frac{\sin[(n+1)\theta - n\theta]}{\cos n\theta\cos(n+1)\theta}\right]$$

$$= -\frac{1}{\sin\theta}\left[\frac{\sin(n+1)\theta\cos n\theta - \cos(n+1)\theta\sin n\theta}{\cos n\theta\cos(n+1)\theta}\right]$$

$$= \frac{-1}{\sin\theta}\left[\frac{\sin(n+1)\theta}{\cos(n+1)\theta} - \frac{\sin n\theta}{\cos n\theta}\right]$$

$$= \frac{-1}{\sin\theta}\left[\tan(n+1)\theta - \tan(n\theta)\right]$$

$$T_n = -\operatorname{cosec}\theta\left[\tan(n+1)\theta - \tan n\theta\right]$$

$n=1$,

$$T_1 = -\operatorname{cosec}\theta\left[\tan 2\theta - \tan\theta\right]$$

$n=2$,

$$T_2 = -\operatorname{cosec}\theta\left[\tan 3\theta - \tan 2\theta\right]$$

$n=3$,

$$T_3 = -\operatorname{cosec}\theta\left[\tan 4\theta - \tan 3\theta\right]$$

$\vdots$

$n = n-1$,

$$T_{n-1} = -\operatorname{cosec}\theta\left[\tan n\theta - \tan(n-1)\theta\right]$$

$n=n$,

$$T_n = -\operatorname{cosec}\theta\left[\tan(n+1)\theta - \tan n\theta\right]$$

$$\therefore S_n = -\operatorname{cosec}\theta\left[\tan(n+1)\theta - \tan\theta\right].$$

7. Find $S_n = \tan^{-1}(1/3) + \tan^{-1}(1/7) + \tan^{-1}(1/13) + \ldots n$ terms

<u>soln:</u>

$$S_n = \tan^{-1}\left[\frac{1}{1+1(2)}\right] + \tan^{-1}\left[\frac{1}{1+2(3)}\right] + \tan^{-1}\left[\frac{1}{1+3(4)}\right] + \ldots$$

$$T_n = \tan^{-1}\left[\frac{1}{1+n(n+1)}\right]$$

Add & sub by n.

$$= \tan^{-1}\left[\frac{1+n-n}{1+n(n+1)}\right]$$

$$= \tan^{-1}\left[\frac{(n+1)-n}{1+n(n+1)}\right]$$

$$\left\{\tan^{-1}A - \tan^{-1}B = \tan^{-1}\left[\frac{A-B}{1+AB}\right]\right\}$$

$$T_n = \tan^{-1}(n+1) - \tan^{-1}(n)$$

$n=1,$

$T_1 = \tan^{-1}(2) - \tan^{-1}(1)$

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$n=2,$

$T_2 = \tan^{-1}(3) - \tan^{-1}(2)$

$n=3,$

$T_3 = \tan^{-1}(4) - \tan^{-1}(3)$

$\vdots$

$n=n-1,$

$T_{n-1} = \tan^{-1} n - \tan^{-1}(n-1)$

$n=n,$

$T_n = \tan^{-1}(n+1) - \tan^{-1}(n)$

$\therefore S_n = \tan^{-1}(n+1) - \tan^{-1}(1).$

8. Find $S_n = \tan^{-1}\left[\frac{1}{1+1+1^2}\right] + \tan^{-1}\left[\frac{1}{1+2+2^2}\right] + \tan^{-1}\left[\frac{1}{1+3+3^2}\right] + \dots n$ term

Soln:

$S_n = \tan^{-1}\left[\frac{1}{1+n+n^2}\right]$

$= \tan^{-1}\left[\frac{1}{1+n(n+1)}\right]$

Add & sub by n.

$= \tan^{-1}\left[\frac{1+n-n}{1+n(n+1)}\right]$

$= \tan^{-1}\left[\frac{(n+1)-n}{1+n(n+1)}\right]$

$\left\{\tan^{-1}A - \tan^{-1}B = \tan^{-1}\left[\frac{A-B}{1-AB}\right]\right\}$

$T_n = \tan^{-1}(n+1) - \tan^{-1} n$

$n=1,$

$T_1 = \tan^{-1}(2) - \tan^{-1}(1)$

$n=2,$

$T_2 = \tan^{-1}(3) - \tan^{-1}(2)$

$n=3,$

$T_3 = \tan^{-1}(4) - \tan^{-1}(3)$

$n = \vdots$

$n = n-1,$

$T_{n-1} = \tan^{-1}(n) - \tan^{-1}(n-1)$

$n=n,$

$T_n = \tan^{-1}(n+1) - \tan^{-1}(n)$

$\therefore S_n = \tan^{-1}(n+1) - \tan^{-1}(1).$

5. Find the $S_n = \tan\theta\sec 2\theta + \tan 2\theta\sec 4\theta + \tan 4\theta \sec 8\theta + \dots$ n term

Soln:

$$S_n = \frac{\sin\theta}{\cos\theta\cos 2\theta} + \frac{\sin 2\theta}{\cos 2\theta\cos 4\theta} + \frac{\sin 4\theta}{\cos 4\theta \cos 8\theta} + \dots + n\text{ term}$$

$$T_n = \frac{\sin(2^{n-1})\theta}{\cos(2^{n-1}\theta)\cos(2^n\theta)}$$

$$= \frac{\sin(2^{n-1})\theta}{\cos(2^{n-1}\theta)\cos 2^n\theta}$$

$\div$ & $\times$ by $2\cos(2^{n-1}\theta)$

$$= \frac{1}{2\cos(2^{n-1}\theta)}\left[\frac{2\sin(2^{n-1}\theta)\cos(2^{n-1}\theta)}{\cos(2^{n-1}\theta)\cos 2^n\theta}\right]$$

$$= \frac{1}{2\cos(2^{n-1})\theta}\left[\frac{\sin(2\times 2^{n-1}\theta)}{\cos(2^{n-1}\theta)\cos 2^n\theta}\right]$$

$$= \frac{1}{2}\sec(2^{n-1})\theta\left[\frac{\sin[(2^n\theta) - (2^{n-1}\theta)]}{\cos(2^{n-1}\theta)\cos 2^n\theta}\right]$$

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$$=\frac{1}{2}\sec(2^{n-1})\theta\left[\frac{\sin 2^n\theta\cos 2^{n-1}\theta-\cos 2^n\theta\sin 2^{n-1}\theta}{\cos(2^{n-1}\theta)\cos(2^n\theta)}\right]$$

$$=\frac{1}{2}\sec(2^{n-1})\theta\left[\frac{\sin 2^n\theta\cos 2^{n-1}\theta}{\cos(2^{n-1}\theta)\cos(2^n\theta)}-\frac{\cos 2^n\theta\sin(2^{n-1}\theta)}{\cos(2^{n-1}\theta)\cos 2^n\theta}\right]$$

$$=\frac{1}{2}\sec(2^{n-1})\theta\left[\frac{\sin 2^n\theta}{\cos 2^n\theta}-\frac{\sin 2^{n-1}\theta}{\cos 2^{n-1}\theta}\right]$$

$$T_n=\frac{1}{2}\sec(2^{n-1})\theta\left[\tan 2^n\theta-\tan 2^{n-1}\theta\right]$$

$n=1$,

$$T_1=\frac{1}{2}\sec(2^{n-1})\theta\left[\tan 2\theta-\tan\theta\right]$$

$n=2$,

$$T_2=\frac{1}{2}\sec(2^{n-1})\theta\left[\tan 4\theta-\tan 2\theta\right]$$

$n=3$,

$$T_3=\frac{1}{2}\sec(2^{n-1})\theta\left[\tan 8\theta-\tan 4\theta\right]$$

$n=n-2$,

$$T_{n-2}=\frac{1}{2}\sec(2^{n-1})\theta\left[\tan 2^{n-2}\theta-\tan 2^{n-3}\theta\right]$$

$n=n-1$,

$$T_{n-1}=\frac{1}{2}\sec(2^{n-1})\theta\left[\tan 2^{n-1}\theta-\tan 2^{n-2}\theta\right]$$

$n=n$,

$$T_n=\frac{1}{2}\sec(2^{n-1})\theta\left[\tan 2^n\theta-\tan 2^{n-1}\theta\right]$$

$$S_n=\frac{1}{2}\sec(2^{n-1})\theta\left[\tan 2^n\theta-\tan\theta\right]$$

9. Find $S_n = \tan x \tan(x+y) + \tan(x+y)\tan(x+2y) + \tan(x+2y)\tan(x+3y) + \dots$ n term

Soln:

$S_n = \tan x \tan(x+y) + \tan(x+y)\tan(x+2y) + \dots$ n terms

$T_n = \tan[x+(n-1)y]\tan(x+ny)$

let $\tan y = \tan[(x+ny) - (x+(n-1)y)]$

$\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A \tan B}$

$\therefore \tan y = \dfrac{\tan(x+ny) - \tan[x+(n-1)y]}{1+\tan(x+ny)\tan[x+(n-1)y]}$

$1+\tan(x+ny)\tan[x+(n-1)y] = \dfrac{1}{\tan y}[\tan(x+ny) - \tan(x+(n-1)y)]$

$1+T_n = \cot y[\tan(x+ny) - \tan(x+(n-1)y)]$

$T_n = \cot y[\tan(x+ny) - \tan(x+(n-1)y)] - 1$

$n=1,$

$T_1 = \cot y[\tan(x+y) - \tan x] - 1$

$n=2,$

$T_2 = \cot y[\tan(x+2y) - \tan(x+y)] - 1$

$n=3,$

$T_3 = \cot y[\tan(x+3y) - \tan(x+2y)] - 1$

$\vdots$

$n=n-1,$

$T_3 = \cot y[\tan(x+(n-1)y) - \tan(x+(n-2)y)] - 1$

$n = n,$

$T_n = \cot y\,[\tan(x+ny) - \tan(x+(n-1)y)] - 1$

$\therefore S_n = \cot y\,[\tan(x+ny) - \tan x] - 1$

10. Find $S_n = \dfrac{\sin 2\theta}{\cos\theta\cos 3\theta} + \dfrac{\sin 4\theta}{\cos 3\theta\cos 5\theta} + \dfrac{\sin 6\theta}{\cos 5\theta\cos 7\theta} + \ldots$ n term

Soln:

$$S_n = \frac{\sin 2\theta}{\cos\theta\cos 3\theta} + \frac{\sin 4\theta}{\cos 3\theta\cos 5\theta} + \ldots \text{ n terms}$$

$$T_n = \frac{\sin 2n\theta}{\cos(2n-1)\theta\,\cos(2n+1)\theta}$$

$\times$ & $\div$ by $2\,\sin\theta$

$$= \frac{1}{2\sin\theta}\left[\frac{2\sin 2n\theta \cdot \sin\theta}{\cos(2n-1)\theta\,\cos(2n+1)\theta}\right]$$

$$= \frac{1}{2}\operatorname{cosec}\theta\left[\frac{\cos(2n-1)\theta - \cos(2n+1)\theta}{\cos(2n-1)\theta\,\cos(2n+1)\theta}\right]$$

$$T_n = \frac{\operatorname{cosec}\theta}{2}\left[\sec(2n+1)\theta - \sec(2n-1)\theta\right].$$

$n = 1,$

$$T_1 = \frac{\operatorname{cosec}\theta}{2}\left[\sec 3\theta - \sec\theta\right]$$

$n = 2,$

$$T_2 = \frac{\operatorname{cosec}\theta}{2}\left[\sec 5\theta - \sec 3\theta\right]$$

$n = 3,$

$$T_3 = \frac{\operatorname{cosec}\theta}{2}\left[\sec 7\theta - \sec 5\theta\right].$$

$2(n-1)+1$

$2n-2+1$

$2n-1$

$n = n-1,$

$$T_{n-1} = \frac{\operatorname{cosec}\theta}{2}\left[\sec(2n-1)\theta - \sec(2n-2)\theta\right]$$

$n = n,$

$$T_n = \frac{\operatorname{cosec}\theta}{2}\left[\sec(2n+1)\theta - \sec(2n-1)\theta\right]$$

$$S_n = \frac{\operatorname{cosec}\theta}{2}\left[\sec(2n+1)\theta - \sec\theta\right].$$

11. Find $S_n = \dfrac{\sin 2\theta}{\sin\theta \sin 3\theta} - \dfrac{\sin 4\theta}{\sin 3\theta \sin 5\theta} + \dfrac{\sin 6\theta}{\sin 5\theta \sin 7\theta}$ ---- n terms

Soln:

$$S_n = \frac{\sin 2\theta}{\sin\theta \sin 3\theta} - \frac{\sin 4\theta}{\sin 3\theta \sin 5\theta} + \dots \text{ n terms}$$

$$T_n = (-1)^{n+1}\frac{\sin 2n\theta}{\sin(2n-1)\theta \sin(2n+1)\theta}$$

$\times E_q \div 2\cos\theta$

$$= \frac{(-1)^{n+1}}{2\cos\theta}\left[\frac{2\sin 2n\theta \,.\, \cos\theta}{\sin(2n-1)\theta \sin(2n+1)\theta}\right]$$

$$= \frac{(-1)^{n+1}}{2}\sec\theta\left[\frac{\sin(2n+1)\theta + \sin(2n-1)\theta}{\sin(2n-1)\theta \sin(2n+1)\theta}\right]$$

$$= \frac{(-1)^{n+1}\sec\theta}{2}\left[\frac{\sin(2n+1)\theta}{\sin(2n-1)\theta \sin(2n+1)\theta} + \frac{\sin(2n-1)\theta}{\sin(2n-1)\theta \sin(2n+1)\theta}\right]$$

$$T_n = \frac{(-1)^{n+1}\sec\theta}{2}\left[\operatorname{cosec}(2n-1)\theta + \cos(2n+1)\theta\right]$$

$n = 1,$

$$T_1 = \frac{(-1)^{1+1}\sec\theta}{2}\left[\operatorname{cosec}\theta \pm \operatorname{cosec}3\theta\right]$$

$n = 2,$

$$T_2 = \frac{(-1)^{2+1}\sec\theta}{2}\left[\operatorname{cosec}3\theta \mp \operatorname{cosec}5\theta\right]$$

$n=3,$

$$T_3 = \frac{\sec\theta}{2}[\operatorname{cosec} 5\theta - \operatorname{cosec} 7\theta]$$

$\vdots$

$n=n-1,$

$$T_{n-1} = \frac{\sec\theta}{2}[\operatorname{cosec}(2n+1)\theta - \operatorname{cosec}(2n-1)\theta]$$

$n=n,$

$$T_n = \frac{\sec\theta}{2}[\operatorname{cosec}(2n-1)\theta - \operatorname{cosec}(2n+1)\theta]$$

$$\therefore S_n = \frac{(-1)^{n+1}\sec\theta}{2}[\operatorname{cosec}\theta + \operatorname{cosec}(2n+1)\theta].$$

Gregory method:

1. Theorem: If $-\frac{1}{4}\pi \le \theta \le \frac{1}{4}\pi$

then $\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \dots\dots$

Proof:

consider $1+i\tan\theta$

$$1+i\tan\theta = 1+i\frac{\sin\theta}{\cos\theta}$$

$$= \frac{\cos\theta + i\sin\theta}{\cos\theta}$$

$$= \sec\theta(\cos\theta + i\sin\theta)$$

$$1+i\tan\theta = \sec\theta . e^{i\theta} \rightarrow (1)$$

Taking log on both sides

$$\log(1+i\tan\theta) = \log(\sec\theta . e^{i\theta})$$

$$= \log(\sec\theta) + \log e^{i\theta}$$

$$\log(1+i\tan\theta) = \log(\sec\theta) + i\theta \rightarrow (2)$$

Since, $-\frac{1}{4}\pi \le \theta \le \frac{1}{4}\pi \Rightarrow |\tan\theta| \le 1$

$\tan(\frac{-\pi}{4}) \le \tan\theta \le \tan(\frac{\pi}{4})$

$-1 \le \tan\theta \le 1$

$\therefore \log(1+i\tan\theta)$ is expanded in logarithmic series.

$$\log(1+i\tan\theta) = (i\tan\theta) - \frac{(i\tan\theta)^2}{2} + \frac{(i\tan\theta)^3}{3} - \frac{(i\tan\theta)^4}{4} + \frac{(i\tan\theta)^5}{5} - \ldots$$

Now sub $\log(1+i\tan\theta) = \log(\sec\theta) + i\theta$

$$\log(\sec\theta) + i\theta = i\tan\theta + \frac{1}{2}\tan^2\theta - \frac{i\tan^3\theta}{3} - \frac{\tan^4\theta}{4} + \frac{i\tan^5\theta}{5} - \ldots$$

Equating imaginary parts

$$\theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{1}{5}\tan^5\theta - \ldots$$

2. Corollary:

If $-1 \le x \le 1$ then $\tan^{-1}(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \ldots$

Proof:

put $x = \tan\theta$ in above thrm

The condition of the theorem is,

$-\frac{1}{4}\pi \le \theta \le \frac{\pi}{4}$

is can be written as

$\tan(\frac{-\pi}{4}) \le \tan\theta \le \tan\pi/4$

$-1 \le \tan\theta \le 1$

Hence, $-1 \le x \le 1$

If $\tan\theta = x \Rightarrow \theta = \tan^{-1}x$

$$\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \ldots$$

$\tan^{-1}(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots$

Note: put $x = 1$,

$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

3. The general form of Gregory's theorem:

Theorem: If $n\pi - \frac{1}{4}\pi \leq \theta \leq n\pi + \frac{1}{4}\pi$

Then, $\theta - n\pi = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \dots$

Proof:

Let $\theta = n\pi + \phi$ where, $\Rightarrow \phi = \theta - n\pi$

$-\frac{1}{4}\pi \leq \phi \leq \frac{1}{4}\pi$

Consider, $1 + i\tan\theta = 1 + i\tan(n\pi + \phi)$

$= 1 + i\tan\phi$

$= 1 + i\frac{\sin\phi}{\cos\phi}$

$= \frac{\cos\phi + i\sin\phi}{\cos\phi}$

$= \sec\phi\,(\cos\theta + i\sin\phi)$

$1 + i\tan\theta = \sec\phi \cdot e^{i\phi}$

Taking log on both sides.

$\log(1 + i\tan\theta) = \log(\sec\phi) + i\phi$

since, $-\frac{1}{4}\pi \leq \phi \leq \frac{1}{4}\pi \Rightarrow |\tan\phi| \leq 1$

$\tan(-\frac{\pi}{4}) \leq \tan\phi \leq \tan\frac{\pi}{4}$

$-1 \leq \tan\phi \leq 1$

$\therefore \log(1 + i\tan\theta)$ is expanded in logarithmic series.

$$\log(1+i\tan\theta) = (i\tan\theta) - \frac{(i\tan\theta)^2}{2} + \frac{(i\tan\theta)^3}{3} - \frac{(i\tan\theta)^4}{4} + \frac{(i\tan\theta)^5}{5} - \dots$$

$$= i\tan\theta + \frac{1}{2}\tan^2\theta - \frac{1}{3}i\tan^3\theta - \frac{1}{4}\tan^4\theta + \frac{1}{5}i\tan^5\theta - \dots$$

$$\log(\sec\varphi) + i\varphi = i\tan\theta + \frac{1}{2}\tan^2\theta - \frac{1}{3}i\tan^3\theta - \frac{1}{4}\tan^4\theta + \frac{1}{5}i\tan^5\theta - \dots$$

Equating imaginary parts

$$\varphi = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \dots$$

$$\because \varphi = \theta - n\pi$$

$$\theta - n\pi = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \dots$$

1. P.T: Gregory Series

Sum of the series: $\left(\frac{2}{3}+\frac{1}{7}\right) - \frac{1}{3}\left(\frac{2}{3^3}+\frac{1}{7^3}\right) + \frac{1}{5}\left(\frac{2}{3^5}+\frac{1}{7^5}\right) - \dots$

$$S_n = \left(\frac{2}{3}+\frac{1}{7}\right) - \frac{1}{3}\left(\frac{2}{3^3}+\frac{1}{7^3}\right) + \frac{1}{5}\left(\frac{2}{3^5}+\frac{1}{7^5}\right) - \dots$$

$$S_n = \left(\frac{2}{3} - \frac{1}{3}\cdot\frac{2}{3^3} + \frac{1}{5}\cdot\frac{2}{3^5} - \dots\right) + \left(\frac{1}{7} - \frac{1}{3}\frac{1}{7^3} + \frac{1}{5}\frac{1}{7^5} - \right)$$

$$= 2\left(\frac{1}{3} - \frac{1}{3}\frac{1}{3^3} + \frac{1}{5}\frac{1}{3^5} \dots\right) + \left(\frac{1}{7} - \frac{1}{3}\frac{1}{7^3} + \frac{1}{5}\frac{1}{7^5} - \right)$$

$$S_n = 2\left(\frac{1}{3} - \frac{1}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{5}\left(\frac{1}{3}\right)^5 - \dots\right) + \left(\frac{1}{7} - \frac{1}{3}\left(\frac{1}{7}\right)^3 + \frac{1}{5}\left(\frac{1}{7}\right)^5 - \dots\right)$$

$$\because \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

$$S_n = 2\tan^{-1}(1/3) + \tan^{-1}(1/7)$$

$$S_n = \tan^{-1}\left[\frac{2(1/3)}{1-(\frac{1}{3})^2}\right] + \tan^{-1}(\tfrac{1}{7}) \quad 2\tan^{-1}x = \left(\frac{2x}{1-x^2}\right)$$

$$= \tan^{-1}\left[\frac{\frac{2}{3}}{\frac{9-1}{9}}\right] + \tan^{-1}(\tfrac{1}{7})$$

$$= \tan^{-1}\left(\frac{2}{3} \times \frac{9}{8}\right) + \tan^{-1}(1/7)$$

$$= \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}(1/7)$$

$$= \tan^{-1}\left[\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}(\frac{1}{7})}\right]$$

$$= \tan^{-1}\left[\frac{\frac{21+4}{28}}{1-\frac{3}{28}}\right]$$

$$= \tan^{-1}\left[\frac{\frac{25}{28}}{\frac{28-3}{28}}\right]$$

$$= \tan^{-1}(1)$$

$$S_n = \pi/4$$

3. P.T: $\frac{\pi}{12} = (1-3^{-1/2}) - \frac{1}{3}(1-3^{-3/2}) + \frac{1}{5}(1-3^{-5/2}) - \dots$

<u>soln</u>:

$$S_n = (1-3^{-1/2}) - \frac{1}{3}(1-3^{-3/2}) + \frac{1}{5}(1-3^{-5/2}) - \dots$$

$$= \left[1-\frac{1}{3}+\frac{1}{5}-\dots\right] - \left[3^{-1/2} - \frac{1}{3}3^{-3/2} + \frac{1}{5}3^{-5/2} - \dots\right]$$

$$= \tan^{-1}(1) - \left[\frac{1}{\sqrt{3}} - \frac{1}{3}(3^{-1/2})^3 + \frac{1}{5}(3^{-1/2})^5 - \dots\right]$$

$$= \pi/4 - \left[\frac{1}{\sqrt{3}} - \frac{1}{3}\cdot\left(\frac{1}{\sqrt{3}}\right)^3 + \frac{1}{5}\left(\frac{1}{\sqrt{3}}\right)^5 - \dots\right]$$

$$S_n = \frac{\pi}{4} - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{\pi}{4} - \frac{\pi}{6}$$

$$= \frac{3\pi - 2\pi}{12}$$

$$S_n = \frac{\pi}{12}.$$

3. P.T : $\pi = 2\sqrt{3}\left[1 - \frac{1}{3^2} + \frac{1}{5}\frac{1}{3^2} - \frac{1}{7}\cdot\frac{1}{3^3} + \cdots\right]$

<u>Soln:</u>

$$S_n = 2\sqrt{3}\left[1 - \frac{1}{3^2} + \frac{1}{5}\frac{1}{3^2} - \frac{1}{7}\cdot\frac{1}{3^3} + \cdots\right]$$

$\times$ & $\div$ by $\sqrt{3}$

$$S_n = \frac{2\sqrt{3}\cdot\sqrt{3}}{\sqrt{3}}\left[1 - \frac{1}{3^2} + \frac{1}{5}\frac{1}{3^2} - \frac{1}{7}\cdot\frac{1}{3^3} + \cdots\right]$$

$$= \frac{6}{\sqrt{3}}\left[1 - \frac{1}{3^2} + \frac{1}{5}\cdot\frac{1}{3^2} - \frac{1}{7}\cdot\frac{1}{3^3} + \cdots\right]$$

$$= 6\left[\frac{1}{\sqrt{3}} - \frac{1}{3^2}\frac{1}{\sqrt{3}} + \frac{1}{5}\cdot\frac{1}{3^2\sqrt{3}} - \cdots\right]$$

$$= 6\left[\frac{1}{\sqrt{3}} - \frac{1}{3\cdot 3\sqrt{3}} + \frac{1}{5\cdot 3\cdot 3\sqrt{3}} - \cdots\right]$$

$$= 6\left[\frac{1}{\sqrt{3}} - \frac{1}{3}\left(\frac{1}{\sqrt{3}}\right)^3 + \frac{1}{5}\left(\frac{1}{\sqrt{3}}\right)^5 - \cdots\right]$$

$$= 6\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= 6\cdot\frac{\pi}{6}$$

$$S_n = \pi.$$

$$6\left[\frac{1}{\sqrt{3}} - \frac{1}{3\cdot(3^{3/2})} + \frac{1}{5}\frac{1}{(3^{5/2})}\right.$$

$$\frac{1}{\sqrt{3}} - \frac{1}{3}\left(\frac{1}{3^{1/2}}\right)^3 + \frac{1}{5}\cdot\left(\frac{1}{3^{1/2}}\right)^5$$

4. If $|x| < \sqrt{2} - 1$ using gregory's series.

P.T: $2\left[x - \frac{x^3}{3} + \frac{x^5}{5} - \dots\right] = \left[\frac{2x}{1-x^2}\right] - \frac{1}{3}\left[\frac{2x}{1-x^2}\right]^3$

$+ \frac{1}{5}\left[\frac{2x}{1-x^2}\right]^5 - \dots$

<u>Soln</u>:

R.H.S =

$\left[\frac{2x}{1-x^2}\right] - \frac{1}{3}\left[\frac{2x}{1-x^2}\right]^3 + \frac{1}{5}\left[\frac{2x}{1-x^2}\right]^5 - \dots$

Put $y = \frac{2x}{1-x^2}$

$= y - \frac{1}{3}y^3 + \frac{1}{5}y^5 - \dots$

$= \tan^{-1}(y)$

$= \tan^{-1}\left[\frac{2x}{1-x^2}\right]$

$= 2\tan^{-1}(x)$

$= 2\left[x - \frac{x^3}{3} + \frac{x^5}{5} - \dots\right]$

$=$ L.H.S

The expansion for $\tan^{-1}(y)$ valid only if $|y| \leq 1$.

If $\left|\frac{2x}{1-x^2}\right| \leq 1$

if $-1 \leq \frac{2x}{1-x^2} \leq 1$

if $-1(1-x^2) \leq 2x$; if $2x \leq 1(1-x^2)$

$-1 + x^2 \leq 2x$; $2x \leq 1 - x^2$

$x^2 - 2x - 1 \leq 0$; $x^2 + 2x - 1 \leq 0$

$\therefore |x| \leq 1$

$\therefore$ These two inequalities are true only for,

if $|x| \leq \sqrt{2} - 1$

Angles in Arithmetic progression:

1. To find the sum of the sines of a series of angles which are in A.P.

Soln:

Let the series be

$$S_n = \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \dots + \sin(\alpha + (n-1)\beta).$$

Multiplying both sides by $2\sin(\beta/2)$,

$$2\sin(\beta/2) S_n = 2\sin\alpha \sin(\beta/2) + 2\sin(\alpha+\beta)\sin(\beta/2) + \dots + 2\sin(\alpha+(n-1)\beta)\sin(\beta/2).$$

W.K.T: $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

$$2\sin(\beta/2) S_n = [\cos(\alpha-\beta/2) - \cos(\alpha+\beta/2)] + [\cos(\alpha+\beta-\beta/2) - \cos(\alpha+\beta+\beta/2)] + \dots + [\cos(\alpha+(n-1)\beta-\beta/2) - \cos(\alpha+(n-1)\beta+\beta/2)]$$

$$= [\cos(\alpha - \tfrac{\beta}{2}) - \cos(\alpha+\beta/2)] + [\cos(\alpha-\beta/2) - \cos(\alpha+\tfrac{3\beta}{2})] + \dots + [\cos(\alpha+(n-\tfrac{3}{2})\beta) - \cos(\alpha+(n-\tfrac{1}{2})\beta)]$$

$$= \cos(\alpha - \tfrac{\beta}{2}) - \cos(\alpha + (n-\tfrac{1}{2})\beta)$$

$$2\sin(\beta/2) S_n = 2\sin[\alpha + (n-1)\tfrac{\beta}{2}] \sin(\tfrac{n\beta}{2})$$

$$= \cancel{\cos[\alpha+(n-1)\beta/2 - n\beta/2]}$$

$$\sin(\beta/2) S_n = \sin[\alpha + (n-1)\beta/2] \sin(n\beta/2)$$

$$S_n = \frac{\sin[\alpha+(n-1)\beta/2] \sin(n\beta/2)}{\sin(\beta/2)}$$

2. To find the sum of the ~~series~~ cosines of a series of angles which are in A.P.

<u>Soln:</u>

Let the series be

$$S_n = \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \dots + \cos(\alpha+(n-1)\beta)$$

$\times$ by $2\sin(\beta/2)$ on both sides

$$2\sin(\tfrac{\beta}{2}) S_n = 2\cos\alpha \sin(\beta/2) + 2\cos(\alpha+\beta)\sin(\beta/2) + \dots +$$
$$+ 2\cos(\alpha+2\beta)\sin(\beta/2) +$$
$$2\cos(\alpha+(n-1)\beta)\sin(\beta/2).$$

W.K.T: $2\cos A \sin B = \sin(A+B) - \sin(A-B)$.

$$2\sin(\beta/2) S_n = \sin(\alpha+\beta/2) - \sin(\alpha-\beta/2) +$$
$$\sin(\alpha+\beta+\beta/2) - \sin(\alpha+\beta-\beta/2) +$$
$$\sin(\alpha+2\beta+\beta/2) - \sin(\alpha+2\beta-\beta/2)$$
$$+ \dots + \sin[\alpha+(n-1)\beta+\beta/2] -$$
$$\sin[\alpha+(n-1)\beta-\beta/2]$$

$$= \sin(\alpha + \beta/2) - \sin(\alpha - \beta/2) +$$
$$\sin(\alpha + \tfrac{3\beta}{2}) - \sin(\alpha + \beta/2) +$$
$$\sin(\alpha + \tfrac{5\beta}{2}) - \sin(\alpha + \tfrac{3\beta}{2}) + \dots +$$
$$\sin(\alpha + (n - \tfrac{1}{2})\beta) - \sin(\alpha + (n - \tfrac{3}{2})\beta)$$
$$= -\sin(\alpha - \beta/2) + \sin[\alpha + (n - \tfrac{1}{2})\beta]$$
$$= \sin[\alpha + (n - \tfrac{1}{2})\beta] - \sin(\alpha - \beta/2)$$
$$2\sin\beta/2 \; S_n = 2\cos(n\beta/2)\cos[\alpha + (n-1)\tfrac{\beta}{2}]$$
$$\sin\beta/2 \; S_n = \cos(\tfrac{n\beta}{2})\cos[\alpha + (n-1)\tfrac{\beta}{2}]$$
$$S_n = \frac{\sin(\tfrac{n\beta}{2})\cos[\alpha + (n-1)\tfrac{\beta}{2}]}{\sin\beta/2}$$

3. Find $S_n = \sin\alpha + \sin 2\alpha + \sin 3\alpha + \dots$ n terms

Soln:

$S_n = \sin\alpha + \sin 2\alpha + \sin 3\alpha + \dots$ n terms.

$\times$ by $2\sin(\alpha/2)$ on both sides

$$2\sin(\alpha/2) S_n = 2\sin\alpha \sin(\alpha/2) + 2\sin 2\alpha \sin(\alpha/2) + 2\sin 3\alpha \sin(\alpha/2) + \dots + 2\sin n\alpha \sin(\alpha/2)$$

W.K.T: $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

$$2\sin(\alpha/2) S_n = \cos(\alpha - \alpha/2) - \cos(\alpha + \alpha/2) +$$
$$\cos(2\alpha - \alpha/2) - \cos(2\alpha + \alpha/2) +$$
$$\cos(3\alpha - \alpha/2) - \cos(3\alpha + \alpha/2) + \dots$$
$$\cos(n\alpha - \alpha/2) - \cos(n\alpha + \alpha/2).$$

$= \cos(\alpha/2) - \cos(\frac{3\alpha}{2}) + \cos(3\alpha/2)$

$- \cos(5\alpha/2) + \cos(5\alpha/2) -$

$\cos(6\alpha/2) + \dots + \cos(\frac{2n\alpha - \alpha}{2})$

$- \cos(\frac{2n\alpha + \alpha}{2})$.

$= \cos(\alpha/2) - \cos(\frac{2n\alpha + \alpha}{2})$

$= \cos(\alpha/2) - \cos(2n+1)\alpha/2$

$2 \sin \alpha/2 \cdot S_n = 2 \sin(n+1)\alpha/2 \cdot \sin(n\alpha/2)$

$$S_n = \frac{\sin(n+1)\alpha/2 \cdot \sin(n\alpha/2)}{\sin \alpha/2}$$

4\. P.T: $\tan n\alpha = \frac{\sin\alpha + \sin 3\alpha + \sin 5\alpha + \dots n \text{ terms}}{\cos\alpha + \cos 3\alpha + \cos 5\alpha + \dots n \text{ terms}}$

Soln:

W.K.T: R.H.S:

$$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \dots = \frac{\sin(\alpha + (n-1)\beta/2)\ \sin(n\beta/2)}{\sin(\beta/2)} \rightarrow ①$$

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \dots = \frac{\cos(\alpha+(n-1)\beta/2)\ \sin(n\beta/2)}{\sin(\beta/2)} \rightarrow ②$$

Gin,

$\sin\alpha + \sin 3\alpha + \sin 5\alpha + \dots = \sin\alpha + \sin(\alpha + 2\alpha) + \sin(\alpha + 4\alpha) + \dots$

$\alpha = \alpha,\ \beta = 2\alpha$ sub in ①

$$① \Rightarrow = \frac{\sin[\alpha + (n-1)\frac{2\alpha}{2}]\ \sin(\frac{n2\alpha}{2})}{\sin(\frac{2\alpha}{2})}$$

$$= \frac{\sin(\alpha + n\alpha - \alpha)\sin(n\alpha)}{\sin\alpha}$$

$$= \frac{\sin(n\alpha)\sin(n\alpha)}{\sin\alpha}$$

$$= \frac{\sin^2(n\alpha)}{\sin\alpha} \rightarrow (3)$$

Sin,

$$\cos\alpha + \cos 3\alpha + \cos 5\alpha + \dots = \cos\alpha + \cos(\alpha + 2\alpha) + \cos(\alpha + 4\alpha) + \dots$$

$\alpha = \alpha$, $\beta = 2\alpha$, sub in (2)

$$(2) \Rightarrow = \frac{\cos\left[\alpha + (n-1)\frac{2\alpha}{2}\right]\sin\left(\frac{n\,2\alpha}{2}\right)}{\sin\left(\frac{2\alpha}{2}\right)}$$

$$= \frac{\cos(\alpha + n\alpha - \alpha)\sin(n\alpha)}{\sin\alpha}$$

$$= \frac{\cos(n\alpha)\sin(n\alpha)}{\sin\alpha} \rightarrow (4)$$

$(3) \div (4)$

$$= \frac{\sin^2(n\alpha)}{\sin\alpha} \times \frac{\sin\alpha}{\cos(n\alpha)\sin(n\alpha)}$$

$$= \frac{\sin(n\alpha)}{\cos(n\alpha)}$$

$$= \tan(n\alpha)$$

$= L.H.S$

5. Find $S_n = \cos^2\alpha + \cos^2(\alpha+\beta) + \cos^2(\alpha+2\beta) + \dots + n$ terms

Hence find $\cos^2\alpha + \cos^2 2\alpha + \cos^2 3\alpha + \dots + n$ terms

Soln:

$$S_n = \cos^2\alpha + \cos^2(\alpha+\beta) + \cos^2(\alpha+2\beta) + \dots + n \text{ terms}$$

$$= \left[\frac{1+\cos 2\alpha}{2}\right] + \left[\frac{1+\cos 2(\alpha+\beta)}{2}\right] + \left[\frac{1+\cos 2(\alpha+2\beta)}{2}\right] + \dots + n \text{ terms}$$

$$= \left[\frac{1}{2} + \frac{1}{2} + \dots + n \text{ terms}\right] + \frac{1}{2}\left[\cos 2\alpha + \cos 2(\alpha+\beta) + \cos 2(\alpha+2\beta) + \dots + n \text{ terms}\right]$$

$$= \frac{n}{2} + \frac{1}{2}\left[\frac{\cos\left[2\alpha + (n-1)\frac{2\beta}{2}\right] \sin\left(\frac{n2\beta}{2}\right)}{\sin 2\beta/2}\right]$$

$$= \frac{n}{2} + \frac{1}{2}\left[\frac{\cos[2\alpha + (n-1)\beta] \sin(n\beta)}{\sin\beta}\right]$$

(ii) $\cos^2\alpha + \cos^2(\alpha+\beta) + \cos^2(\alpha+2\beta) + \dots + n$ terms

$$= \frac{n}{2} + \frac{1}{2}\left[\frac{\cos[2\alpha + (n-1)\beta] \sin n\beta}{\sin\beta}\right] \rightarrow (1)$$

put $\beta = \alpha$ in eqn (1)

$\therefore \cos^2\alpha + \cos^2 2\alpha + \cos^2 3\alpha + \dots + n$ terms.

$$= \frac{n}{2} + \frac{1}{2}\left[\frac{\cos(2\alpha + n\alpha - \alpha) \sin n\alpha}{\sin\alpha}\right]$$

$$= \frac{n}{2} + \frac{1}{2}\left[\frac{\cos(\alpha + n\alpha) \sin n\alpha}{\sin\alpha}\right].$$

6. Find $S_n = \cos^3\alpha + \cos^3 2\alpha + \cos^3 3\alpha + \dots + n$ terms

Soln:

$$S_n = \frac{1}{4}[\cos 3\alpha + 3\cos\alpha] + \frac{1}{4}[\cos 6\alpha + 3\cos 2\alpha] + \frac{1}{4}[\cos 9\alpha + 3\cos 3\alpha] + \dots + n \text{ term}$$

$$= \frac{1}{4}[\cos 3\alpha + \cos 6\alpha + \cos 9\alpha + \dots + n \text{ terms}] + \frac{3}{4}[\cos\alpha + \cos 2\alpha + \cos 3\alpha + \dots + n \text{ terms}]$$

$\alpha = 3\alpha$, $\beta = 3\alpha$

$$= \frac{1}{4}\left[\frac{\cos\left[3\alpha + (n-1)\frac{3\alpha}{2}\right]\sin\left[\frac{n3\alpha}{2}\right]}{\sin\left(\frac{3\alpha}{2}\right)}\right] + \frac{3}{4}\left[\frac{\cos \alpha + (n-1)\alpha/2 \,\sin\left(\frac{n\alpha}{2}\right)}{\sin(\alpha/2)}\right]$$

$$= \frac{1}{4}\left[\frac{\cos\left(3\alpha + \frac{3n\alpha - 3\alpha}{2}\right)\sin\left(\frac{3n\alpha}{2}\right)}{\sin(3\alpha/2)}\right] + \frac{3}{4}\left[\frac{\cos\left(\alpha + \frac{n\alpha - \alpha}{2}\right)\sin\left(\frac{n\alpha}{2}\right)}{\sin(\alpha/2)}\right]$$

$$= \frac{1}{4}\left[\frac{\cos\left(\frac{3\alpha + 3n\alpha}{2}\right)\sin\left(\frac{3n\alpha}{2}\right)}{\sin(3\alpha/2)}\right] + \frac{3}{4}\left[\frac{\cos\left(\frac{\alpha + n\alpha}{2}\right)\sin(n\alpha/2)}{\sin(\alpha/2)}\right]$$

$$= \frac{1}{4}\left[\frac{\cos\left[\left(\frac{n+1}{2}\right)3\alpha\right]\sin\frac{3n\alpha}{2}}{\sin(3\alpha/2)}\right] + \frac{3}{4}\left[\frac{\cos\left[\left(\frac{n+1}{2}\right)\alpha\right]\sin\left(\frac{n\alpha}{2}\right)}{\sin(\alpha/2)}\right]$$

7. Find $S_n = \cos\alpha\cos 3\alpha + \cos 3\alpha\cos 5\alpha + \cos 5\alpha\cos 7\alpha + \dots + n$ terms

Soln:

$$S_n = \frac{1}{2}\left[2\cos\alpha\cos 3\alpha + 2\cos 3\alpha\cos 5\alpha + 2\cos 5\alpha\cos 7\alpha + \dots + n \text{ terms}\right]$$

WKT $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$$S_n = \frac{1}{2}\left[(\cos 4\alpha + \cos 2\alpha) + (\cos 8\alpha + \cos 2\alpha) + (\cos 12\alpha + \cos 2\alpha) + \dots + n \text{ terms}\right]$$

$$= \frac{1}{2}\left\{\left[\frac{\cancel{\sin\left(\frac{n4\alpha}{2}\right)}\cos\left[4\alpha + (n-1)\frac{4\alpha}{2}\right]\sin\left(\frac{4n\alpha}{2}\right)}{\sin\left(\frac{4\alpha}{2}\right)}\right] + \left(\frac{n}{2}\cos 2\alpha\right)\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{\cos(4\alpha + (n-1)2\alpha)\sin(2n\alpha)}{\sin 2\alpha}\right] + \left(\frac{n}{2}\cos 2\alpha\right)\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{\cos(2\alpha + 2n\alpha)\sin(2n\alpha)}{\sin 2\alpha}\right] + (n\cos 2\alpha)\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{\cos(n+1)2\alpha\ \sin(2n\alpha)}{\sin 2\alpha}\right] + (n\cos 2\alpha)\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{2\cos(n+1)\alpha\ \sin(2n\alpha)}{\sin 2\alpha}\right] + (n\cos 2\alpha)\right\}$$

5. Find $S_n = \sin\alpha - \sin 2\alpha + \sin 3\alpha - \sin 4\alpha + \dots$ n terms

Soln:

W.K.T : $\sin(\pi+2\alpha) = -\sin 2\alpha$

$\sin(2\pi+3\alpha) = \sin 3\alpha$

$\sin(3\pi+4\alpha) = -\sin 4\alpha$

$\sin(4\pi+5\alpha) = \sin 5\alpha$

$\sin(5\pi+6\alpha) = -\sin 6\alpha$.

By using the above,

$$S_n = \sin\alpha + \sin(\pi+2\alpha) + \sin 3\alpha + \sin(3\pi+4\alpha) + \dots$$

$$= \sin\alpha + \sin[\alpha+(\alpha+\pi)] + \sin[\alpha+2(\alpha+\pi)] + \sin[\alpha+3(\alpha+\pi)] + \dots$$

$\therefore \alpha = \alpha,\ \beta = \alpha+\pi$

$$= \frac{\sin\left[\alpha + \frac{(n-1)(\alpha+\pi)}{2}\right]\sin\left(\frac{n(\alpha+\pi)}{2}\right)}{\sin\left(\frac{\alpha+\pi}{2}\right)}$$

$$= \frac{\sin\left[\frac{2\alpha+n\alpha+n\pi-\alpha-\pi}{2}\right]\sin\left[\frac{n(\alpha+\pi)}{2}\right]}{\sin\left(\frac{\alpha+\pi}{2}\right)}$$

$$= \frac{\sin\left[\frac{\alpha(n+1)+\pi(n-1)}{2}\right]\sin\left[\frac{n(\alpha+\pi)}{2}\right]}{\sin\left(\frac{\alpha+\pi}{2}\right)}$$

$$= \frac{\sin\left[\alpha + \frac{(n-1)(\alpha+\pi)}{2}\right]\sin\left(\frac{n(\alpha+\pi)}{2}\right)}{\cos\left(\frac{\alpha}{2}\right)}$$

$\sin\left(\frac{\pi}{2}+\frac{\alpha}{2}\right)$