## UNIT- IV

# Hyperbolic function

**Definition :**

The hyperbolic functions are defined by

i) $\sinh x = \frac{e^{x} - e^{-x}}{2} \Rightarrow \operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^{x} - e^{-x}}$

ii) $\cosh x = \frac{e^{x} + e^{-x}}{2} \Rightarrow \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^{x} + e^{-x}}$

iii) $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{x} - e^{-x}}{2} \Big/ \frac{e^{x} + e^{-x}}{2} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} \Rightarrow$

$\coth x = \frac{1}{\tanh x} = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$

**Result : 1**

$\cosh^2 x - \sinh^2 x = 1$

**Soln**

$$L.H.S \Rightarrow \cosh^2 x - \sinh^2 x = \left[\frac{e^{x} + e^{-x}}{2}\right]^2 - \left[\frac{e^{x} - e^{-x}}{2}\right]^2$$

$$= \frac{(e^{x} + e^{-x})^2}{4} - \frac{(e^{x} - e^{-x})^2}{4}$$

$$= \frac{1}{4}\left[(e^{x} + e^{-x})^2 - (e^{x} - e^{-x})^2\right]$$

$$= \frac{1}{4}\left[(e^{2x} + e^{-2x} + 2e^{x}.e^{-x}) - (e^{2x} + e^{-2x} - 2e^{x}e^{-x})\right]$$

$$= \frac{1}{4}\left[e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2\right]$$

$$= \frac{1}{4}(4)$$

$$= 1$$

$$= R.H.S$$

Result : 2

$\sinh 2x = 2 \sinh x \cosh x$

R.H.S :

$$2\sinh x \cosh x = 2\left[\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right)\right]$$

$(a+b)(a-b) = a^2 - b^2$

$$= \frac{2}{4}\left[e^{2x} - e^{-2x}\right]$$

$$= \frac{1}{2}\left(e^{2x} - e^{-2x}\right)$$

$$= \sinh 2x$$

Result : 3

$\cosh^2 x + \sinh^2 x = \cosh 2x .$

L.H.S :

$$\cosh^2 x + \sinh^2 x = \left[\frac{e^x + e^{-x}}{2}\right]^2 + \left[\frac{e^x - e^{-x}}{2}\right]^2$$

$$= \frac{(e^x + e^{-x})^2}{4} + \frac{(e^x - e^{-x})^2}{4}$$

$$= \frac{1}{4}\left[(e^x + e^{-x})^2 + (e^x - e^{-x})^2\right]$$

$$= \frac{1}{4}\left[(e^{2x} + e^{-2x} + 2 \cdot e^x \cdot e^{-x}) + (e^{2x} + e^{-2x} - 2e^x e^{-x})\right]$$

$$= \frac{1}{4}\left[(e^{2x} + e^{-2x} + 2) + (e^{2x} + e^{-2x} - 2)\right]$$

$$= \frac{1}{4}\left[e^{2x} + e^{-2x} + 2 + e^{2x} + e^{-2x} - 2\right]$$

$$= \frac{1}{4}\left[e^{2x} + e^{-2x} + e^{2x} + e^{-2x}\right]$$

$$= \frac{1}{4}\left(2e^{2x} + 2e^{-2x}\right)$$

$= \frac{1}{4} \times 2 \, (e^{2x} + e^{-2x})$

$= \frac{1}{2} (e^{2x} + e^{-2x})$

$= \cosh 2x$

$=$ R.H.S

Result : 4

$\cosh 2x = 2\cosh^2 x - 1$

R.H.S :

$2\cosh^2 x - 1 = 2\left(\frac{e^x + e^{-x}}{2}\right)^2 - 1$

$\cancel{= e^x + e^{-x} - 1}$

$= 2 \frac{(e^x + e^{-x})^2}{4} - 1$

$= \frac{1}{2} (e^x + e^{-x})^2 - 1$

$= \frac{1}{2} (e^{2x} + e^{-2x} + 2e^x \cdot e^{-x}) - 1$

$= \frac{1}{2} (e^{2x} + e^{-2x} + 2) - 1$

$= \cancel{\frac{1}{2}} \left(\frac{e^{2x} + e^{-2x} + 2}{2}\right) - 1$

$= \frac{e^{2x} + e^{-2x} + 2 - 2}{2}$

$= \frac{e^{2x} + e^{-2x}}{2}$

$= \cosh 2x$

$=$ L.H.S

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Result: 5

$$\cosh 2x = 1 + 2\sinh^2 x$$

$$\cosh^2 x = \left(\frac{\cosh 2x + 1}{2}\right)$$

$$\sinh^2 x = \left(\frac{\cosh 2x - 1}{2}\right)$$

NOTE:

Immediate consequence of definition.

1. $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

2. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

3. $\cosh 0 = 1$

   $\sinh 0 = 0$

4. W.K.T

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

Relation between hyperbolic functions and circular trigonometric functions:

Theorem: (i) $\sin(ix) = i\sinh x$

(ii) $\cos(ix) = \cosh x$

(iii) $\tan(ix) = i\tanh x$

Proof of (i)

i) W.K.T. $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$

Put $\theta = ix$

$$\sin(ix) = (ix) - \frac{(ix)^3}{3!} + \frac{(ix)^5}{5!} - \frac{(ix)^7}{7!} + \cdots$$

$$= ix - \frac{i^3x^3}{3!} + \frac{i^5x^5}{5!} - \frac{i^7x^7}{7!} + \cdots$$

$$= ix + \frac{ix^3}{3!} + \frac{ix^5}{5!} + \frac{ix^7}{7!} + \cdots$$

$$= i\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\right)$$

$$\sin(ix) = i \sinh x$$

Proof of (ii)

ii) W.K.T $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$

Put $\theta = ix$

$$\cos(ix) = 1 - \frac{(ix)^2}{2!} + \frac{(ix)^4}{4!} - \frac{(ix)^6}{6!} + \cdots$$

$$= 1 - \frac{i^2x^2}{2!} + \frac{i^4x^4}{4!} - \frac{i^6x^6}{6!} + \cdots$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

$$\cos(ix) = \cosh x$$

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Proof of (iii)

(iii) $\tan(ix) = \dfrac{\sin(ix)}{\cos(ix)}$

$= \dfrac{i\sinh x}{\cosh x}$

$\tan(ix) = i\tanh x$

Example: 1

~~Prov~~ Corresponding to the formula $\cos^2\theta + \sin^2\theta = 1$

$\cos^2\theta + \sin^2\theta = 1$

Put $\theta = ix$

$\cos^2(ix) + \sin^2(ix) = 1$

$[\cos(ix)]^2 + [\sin(ix)]^2 = 1$

$[\cosh x]^2 + [i\sinh x]^2 = 1$

$\cosh^2 x - \sinh^2 x = 1$

Example: 2

consider $\cos(A+B) = \cos A\cos B - \sin A\sin B$

Put $A = ix$

$B = iy$

$\cos(ix+iy) = \cos(ix)\cos(iy) - \sin(ix)\sin(iy)$

$\cos i(x+y) = \cosh x.\cosh y - i\sinh x.i\sinh y$

$\cosh(x+y) = \cosh x.\cosh y + \sinh x.\sinh y$

## Inverse hyperbolic function:

**Definition:**

Consider the funtion $y = \sin hx$. This is a 1-1 onto map from $R \to R$ $\therefore$ Given any $y \in R$ there exists unique $x$ such that $\sin hx = y$

we define $x = \sin h^{-1} y$.

$|||^{ly}$ $y = \cos hx$ is a map from $R \to [1, \infty)$. Both $x$ and $-x$ have the same image under $\cos hx$. Hence given by any $y \in [1, \infty)$ we can find a unique $x$ such that $\cos hx = y$

We define $x = \cos h^{-1}(y)$ and $x$ is called the principal value of $\cos h^{-1}(y)$.

$\therefore$ The function $y = \tan hx$ is a map from $R \to (-1, 1)$ given any $y \in R$ $\exists$ unique $x$ such that $\tan hx = y$

We define $x = \tan h^{-1}(y)$.

**Theorems:**

(i) $\sin h^{-1}(x) = \log_e (x + \sqrt{x^2+1})$

(ii) $\cos h^{-1}(x) = \log_e (x + \sqrt{x^2-1})$

(iii) $\tan h^{-1}(x) = \frac{1}{2} \log \left(\frac{1+x}{1-x}\right)$

Proof:

(i) Let $y = \sinh^{-1}(x)$

$x = \sinh y$

$x = \frac{e^{y} - e^{-y}}{2}$

$\times$ both sides $e^{y}$

$xe^{y} = e^{y}\left[\frac{e^{y} - e^{-y}}{2}\right]$

$2xe^{y} = e^{2y} - e^{-y}e^{y}$

$2xe^{y} = e^{2y} - 1$

$e^{2y} - 2xe^{y} - 1 = 0$

$a = 1,\ b = -2x,\ c = -1$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$e^{y} = \frac{2x \pm \sqrt{4x^2 - 4(1)(-1)}}{2}$

$e^{y} = \frac{2x \pm \sqrt{4x^2 + 4}}{2}$

$e^{y} = \frac{2x \pm 2\sqrt{x^2+1}}{2}$

$e^{y} = x \pm \sqrt{x^2+1}$

since $e^{y}$ is always +ve

$e^{y} = x + \sqrt{x^2+1}$

Taking log on both sides,

$y = \log_e (x + \sqrt{x^2+1})$.

Proof:

(ii) Let $y = \cosh^{-1}(x)$

$x = \cosh y$

$x = \frac{e^y + e^{-y}}{2}$

$\times$ both sides $e^y$

$xe^y = e^y\left[\frac{e^y + e^{-y}}{2}\right]$

$2xe^y = e^{2y} + 1$

$e^{2y} - 2xe^y + 1 = 0$

$a = 1, \quad b = -2x, \quad c = 1$

$x = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$

$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2}$

$e^y = \frac{2x \pm 2\sqrt{x^2-1}}{2}$

$e^y = x \pm \sqrt{x^2-1}$

Since $e^y$ is always +ve

$e^y = x + \sqrt{x^2-1}$

Taking log on both sides,

$y = \log_e(x + \sqrt{x^2-1})$.

Proof:

(iii) Let $y = \tan h^{-1}(x)$

$x = \tan h\, y$

$$x = \frac{e^{y} - e^{-y}}{e^{y} + e^{-y}}$$

$$x(e^{y} + e^{-y}) = e^{y} - e^{-y}$$

$$xe^{y} + xe^{-y} = e^{y} - e^{-y}$$

$$xe^{y} - e^{y} = -e^{-y} - xe^{-y}$$

$$e^{y}(x-1) = -e^{-y}(1+x)$$

$$-e^{y}(1-x) = -e^{-y}(1+x)$$

$$e^{y}(1-x) = e^{-y}(1+x)$$

$$\frac{e^{y}}{e^{-y}}(1-x) = (1+x)$$

$$e^{y}.e^{y} = \frac{(1+x)}{(1-x)}$$

$$e^{2y} = \frac{1+x}{1-x}$$

Taking log on both sides

$$2y = \log\left(\frac{1+x}{1-x}\right)$$

$$y = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

1. P.T $(\cosh x - \sinh x)^n = \cosh nx - \sinh nx$

Soln:

L.H.S: $(\cosh x - \sinh x)^n = [\cos(ix) - i\sin(ix)]^n$

$= [\cos n(ix) - i\sin n(ix)]$

by De-Moiver's theorm

$= [\cos i(nx) - i\sin i(nx)]$

$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$

Hence proved L.H.S = R.H.S

2. P.T $\dfrac{1+\tanh x}{1-\tanh x} = \cosh 2x + \sinh 2x$

Soln:

L.H.S: $\dfrac{1+\tanh x}{1-\tanh x} = \dfrac{1+\dfrac{\sinh x}{\cosh x}}{1-\dfrac{\sinh x}{\cosh x}}$

$= \dfrac{\cosh x + \sinh x}{\cosh x} \times \dfrac{\cosh x}{\cosh x - \sinh x}$

$= \dfrac{\cosh x + \sinh x}{\cosh x - \sinh x}$

$= \dfrac{\cosh x + \sinh x}{\cosh x - \sinh x} \times \dfrac{\cosh x + \sinh x}{\cosh x + \sinh x}$

$= \dfrac{(\cosh x + \sinh x)^2}{\cosh^2 x - \sinh^2 x}$

$2\sin\theta\cos\theta = \sin 2\theta$

$\cosh^2 x + \sinh^2 x = \cosh 2x$

$\cosh^2 x - \sinh^2 x =$

$= \dfrac{[\cosh^2 x + \sinh^2 x + 2\sinh x\cosh x]}{1}$

$= \cosh 2x + \sinh 2x$

$\dfrac{1+\tanh x}{1-\tanh x} =$

Hence proved

L.H.S = R.H.S

3. If $\cos(x+iy) = \cos\theta + i\sin\theta$ P.T

$\cos 2x + \cosh 2y = 2$

Soln:

$\cos(x+iy) = \cos\theta + i\sin\theta$

$\cos(x)\cos(iy) - \sin(x)\sin(iy) = \cos\theta + i\sin\theta$

$\cos x \cosh y - i\sin x \sinh y = \cos\theta + i\sin\theta.$

Equating real & imaginary terms

$\cos x \cosh y = \cos\theta \rightarrow (1)$

$-\sin x \sinh y = \sin\theta \rightarrow (2)$

$(1) \Rightarrow \cos^2\theta = \cos^2 x \cosh^2 y$

$(2) \Rightarrow \sin^2\theta = \sin^2 x \sinh^2 y$

$(1)+(2)$

$\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y = \cos^2\theta + \sin^2\theta$

$\cos^2 x \cosh^2 y + (1-\cos^2 x)\sinh^2 y = 1$

$\cos^2 x \cosh^2 y + \sinh^2 y - \cos^2 x \sinh^2 y = 1$

$\cos^2 x (\cosh^2 y - \sinh^2 y) + \sinh^2 y = 1$

$\cos^2 x + \sin^2 y = 1$

$\frac{1+\cos 2x}{2} + \frac{1}{2}(\cosh 2y - 1) = 1$

$1 + \cos 2x + \cosh 2y - 1 = 2$

$\cos 2x + \cosh 2y = 2$

Hence proved.

4. If $\sin(\theta+i\psi) = \tan\alpha + i\sec\alpha$ P.T

$\cos 2\theta \cosh 2y = 3$

Soln:

$\sin(\theta+i\psi) = \tan\alpha + i\sec\alpha$

$\sin\theta\cos(i\psi) + \cos\theta\sin(i\psi) = \tan\alpha + i\sec\alpha$

$\sin\theta\cosh\psi + i\cos\theta\sinh\psi = \tan\alpha + i\sec\alpha$

Equating real & imaginary terms.

$\sin\theta\cosh\psi = \tan\alpha \rightarrow ①$

$\cos\theta\sinh\psi = \sec\alpha \rightarrow ②$

Using this term $1+\tan^2\alpha = \sec^2\alpha$

$1+\sin^2\theta\cosh^2\psi = \cos^2\theta\sinh^2\psi$

$1+(1-\cos^2\theta)\cosh^2\psi = \cos^2\theta\sinh^2\psi$

$1+\cosh^2\psi - \cos^2\theta\cosh^2\psi = \cos^2\theta\sinh^2\psi$

$\cancel{1+\cosh^2\psi\,(1-\cos^2\theta)}$

$1+\cosh^2\psi - \cos^2\theta\cosh^2\psi - \cos^2\theta\sinh^2\psi = 0$

$1+\cosh^2\psi - \cos^2\theta(\cosh^2\psi + \sinh^2\psi) = 0$

$1+\cosh^2\psi - \cos^2\theta(\cosh 2\psi) = 0$

$1+\left(\frac{1+\cosh 2\psi}{2}\right) - \left(\frac{1+\cos 2\theta}{2}\right)\cosh 2\psi = 0$

$2+1+\cosh 2\psi - \cosh 2\psi - \cos 2\theta\cosh 2\psi = 0$

$3 - \cos 2\theta\cosh 2\psi = 0$

$\cos 2\theta\cosh 2\psi = 3$

Hence proved.

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5. If $\tan(a+ib) = x+iy$

P.T $\frac{x}{y} = \frac{\sin 2a}{\sinh 2b}$

Soln:

$x+iy = \tan(a+ib)$

$$x+iy = \frac{\sin(a+ib)}{\cos(a+ib)}$$

$$= \frac{\sin(a+ib)}{\cos(a+ib)} \times \frac{\cos(a-ib)}{\cos(a-ib)}$$

$$x+iy = \frac{2}{2}\,\frac{\sin(a+ib)\cos(a-ib)}{\cos(a+ib)\cos(a-ib)}$$

For understanding purpose.

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

By adding these two we get,

$\sin(A+B) + \sin(A-B) = 2\sin A\cos B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A.\cos B + \sin A\sin B$

By adding these two we get,

$\cos(A+B) + \cos(A-B) = 2\cos A\cos B$

$$\therefore x+iy = \frac{\sin(a+ib+a-ib) + \sin(a+ib-a+ib)}{\cos(a+ib+a-ib) + \cos(a+ib-a+ib)}$$

$$= \frac{\sin 2a + \sin(2ib)}{\cos 2a + \cos(2ib)}$$

$$x+iy = \frac{\sin 2a + i\sinh 2b}{\cos 2a + \cosh 2b}$$

$$x+iy = \frac{\sin 2a}{\cos 2a + \cosh 2b} + \frac{i \sinh 2b}{\cos 2a + \cosh 2b}$$

Equating the co-efficient of real & imaginary.

$$x = \frac{\sin 2a}{\cos 2a + \cosh 2b} \rightarrow ①$$

$$y = \frac{\sinh 2b}{\cos 2a + \cosh 2b} \rightarrow ②$$

$① \div ②$

$$\frac{x}{y} = \frac{\sin 2a}{\cos 2a + \cosh 2b} \times \frac{\cos 2a + \cosh 2b}{\sinh 2b}$$

$$\frac{x}{y} = \frac{\sin 2a}{\sinh 2b}$$

Hence proved.

Q. If $x+iy = \sin(A+iB)$. P.T $\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1$

Soln:

$x+iy = \sin(A+iB)$

$= \sin A \cos(iB) + \cos A \sin(iB)$

$x+iy = \sin A \cosh B + i \cos A \sinh B$

Equating the co-efficient of real & imaginary

$x = \sin A \cosh B$. $\quad y = \cos A \sinh B$

$\frac{x}{\sin A} = \cosh B \rightarrow ①$ $\quad \frac{y}{\cos A} = \sinh B \rightarrow ②$

squaring on both sides

$\frac{x^2}{\sin^2 A} = \cosh^2 B \rightarrow ③$ $\quad \frac{y^2}{\cos^2 A} = \sinh^2 B \rightarrow ④$

③ - ④

$$\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = \cosh^2 B - \sinh^2 B$$

$\cosh^2 B - \sinh^2 B = 1$

$$\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1$$

Hence proved.

7. ~~If~~

$\tan A = \tan\alpha \tanh\beta$, $\tan B = \cot\alpha \tanh\beta$

prove that $\tan(A+B) = \sinh 2\beta \operatorname{cosec} 2\alpha$

Soln:

Given $\tan A = \tan\alpha \tanh\beta$

$\tan B = \cot\alpha \tanh\beta$

W.K.T

$$\tan(A+B) = \frac{\tan A + \tanh\beta}{1 - \tan A \tanh\beta}$$

$$= \frac{\tan\alpha\tanh\beta + \cot\alpha \tanh\beta}{1 - [(\tan\alpha\tanh\beta)(\cot\alpha \tanh\beta)]}$$

$$= \frac{\tanh\beta[\tan\alpha + \cot\alpha]}{1 - ([\tan\alpha \cdot \tanh\beta][1/\tan\alpha \tanh\beta])}$$

$$= \frac{\tanh\beta[\sin\alpha/\cos\alpha + \cos\alpha/\sin\alpha]}{1 - \tan^2 h\beta}$$

$$= \frac{\tanh\beta\left[\frac{\sin^2\alpha + \cos^2\alpha}{\sin\alpha\cos\alpha}\right]}{1 - \frac{\sin^2 h\beta}{\cos^2 h\beta}}$$

$$= \frac{\tanh\beta[1/\sin\alpha\cos\alpha]}{\frac{\cos^2 h\beta - \sin^2 h\beta}{\cos^2 h\beta}}$$

$$= \frac{\tanh\beta}{\sin\alpha\cos\alpha} \times \frac{\cos^2 h\beta}{1}$$

$$= \frac{\tanh\beta}{\sin\alpha\cos\alpha} \times \cos^2 h\beta$$

$$= \frac{\frac{\sin h\beta}{\cos h\beta} \times \cos^2 h\beta}{\sin\alpha\cos\alpha}$$

$$= \frac{\sinh\beta\cosh\beta}{\sin\alpha\cos\alpha}$$

Now $\times$ and $\div 2$,

$$= \frac{2\sinh\beta\cosh\beta}{2\sin\alpha\cos\alpha}$$

$$= \frac{\sin 2h\beta}{\sin 2\alpha}$$

$\tan(A+B) = \sin 2h\beta \operatorname{cosec} 2\alpha$

Hence proved.

8. $\sin h2x = \frac{2\tanh x}{1-\tan^2 hx}$

Soln:

W.K.T $\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$

put $\theta = ix$

$$\sin 2(ix) = \frac{2\tan(ix)}{1+\tan^2(ix)}$$

$$\sin i\,2hx = \frac{2i\tan hx}{1-\tan^2 hx}$$

$$\sin 2hx = \frac{2\tanh x}{1-\tan^2 hx}$$

Hence proved.

9. $\cos h2x = \frac{1+\tan^2 hx}{1-\tan^2 hx}$

Soln:

W.K.T $\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

put $\theta = ix$

$$\cos 2(ix) = \frac{1-\tan^2(ix)}{1+\tan^2(ix)}$$

$$\cos 2hx = \frac{1-i^2\tan^2 hx}{1+i^2\tan^2 hx}$$

$$\cos 2hx = \frac{1+\tan^2 hx}{1-\tan^2 hx}$$

10. $\tan(x+iy)$ seperate into real & imaginary part:

Soln:

$$\tan(x+iy) = \frac{\sin(x+iy)}{\cos(x+iy)}$$

$$= \frac{\sin(x+iy)}{\cos(x+iy)} \times \frac{\cos(x-iy)}{\cos(x-iy)}$$

$$= \frac{\sin(x+iy)\cos(x-iy)}{\cos(x+iy)\cos(x-iy)}$$

$$= \frac{\sin(x+iy+x-iy)+\sin(x+iy-x+iy)}{\cos(x+iy+x-iy)+\cos(x+iy-x+iy)}$$

$$= \frac{\sin(2x)+\sin(2iy)}{\cos(2x)+\cos(2iy)}$$

$$= \frac{\sin 2x + i\sin 2hy}{\cos 2x + \cos 2hy}$$

Equating the real & imaginary part

$x = \frac{\sin 2x}{\cos 2x + \cos 2hy}$ ; $y = \frac{\sin 2hy}{\cos 2x + \cos 2hy}$

11. P.T $\tanh 3x = \dfrac{3\tanh x + \tan^3 hx}{1+3\tan^2 hx}$

Soln:

W.K.T: $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

Put $\theta = ix$

$$\tan 3(ix) = \frac{3\tan(ix) - \tan^3(ix)}{1-3\tan^2(ix)}$$

$$i\tan 3hx = \frac{3i\tanh x - i^3\tan^3 hx}{1-3i^2\tan^2 hx}$$

$$i\tanh 3x = \frac{3i\tanh x + i\tan^3 hx}{1+3\tan^2 hx}$$

$$i(\tanh 3x) = \frac{i(3\tanh x + \tan^3 hx)}{1+3\tan^2 hx}$$

$$\tanh 3x = \frac{3\tanh x + \tan^3 hx}{1+3\tan^2 hx}$$

12. To find real & imaginary parts of $\cosh(x+iy)$

Soln:

$$\cosh(x+iy) = \cos i(x+iy)$$
$$= \cos(ix + i^2 y)$$
$$= \cos(ix - y)$$
$$= \cos(ix)\cos y + \sin(ix)\sin y$$
$$= \cosh x\cos y + i\sinh x\sin y$$

Real part $= \cosh x\cos y$

Imaginary part $= \sinh x\sin y$.

13. To find real & imaginary parts of $\cosh(1+i)$

Soln:

$\cosh(1+i) = \cos i(1+i)$

$= \cos(i+i^2)$

$= \cos(i-1)$

$= \cos i \cos 1 + \sin i \sin 1$

$= \cosh \cos + \sinh \sin$

Real part $= \cosh \cos$

Imaginary part $= \sin h \sin$

14. To find real and imaginary of $\sec(x+iy)$

Soln:

$$\sec(x+iy) = \frac{1}{\cos(x+iy)}$$

$$= \frac{1}{\cos(x+iy)} \times \frac{\cos(x-iy)}{\cos(x-iy)}$$

$$= \frac{2}{2}\frac{\cos(x-iy)}{\cos(x+iy)\cos(x-iy)}$$

$$= \frac{2[\cos x \cos(iy) + \sin x \sin(iy)]}{\cos(x+iy+x-iy) + \cos(x+iy-x+iy)}$$

$$= \frac{2[\cos x \cosh y + i \sin x \sinh y]}{\cos(2x) + \cos(2iy)}$$

$$= \frac{2\cos x \cosh y}{\cos(2x)+\cos(hiy)} + \frac{i \sin x \sinh y}{\cos(2x)+\cos(hiy)}$$

Real part $x = \frac{2\cos x \cosh y}{\cos(2x)\cos(hy)}$

Imaginar part $y = \frac{\sin x \sinh y}{\cos(2x)+\cos(hy)}$

15. To find real & imaginary part of $\cot(x+iy)$

**Soln:**

$$\cot(x+iy) = \frac{\cos(x+iy)}{\sin(x+iy)}$$

$$= \frac{\cos(x+iy)}{\sin(x+iy)} \times \frac{\sin(x-iy)}{\sin(x-iy)}$$

$$= \frac{2}{2}\,\frac{\cos(x+iy)\sin(x-iy)}{\sin(x+iy)\sin(x-iy)}$$

$$= \frac{\sin(x+iy+x-iy) + \sin(x+iy-x+iy)}{\cos(x+iy-x+iy) - \cos(x+iy+x-iy)\ \cancel{\sin(x+iy)\sin(x-iy)}}$$

$$= \frac{\sin(2x) + \sin(2iy)}{\cos(2iy) - \cos 2x\ \cancel{\sin(x+iy)\sin(x-iy)}}$$

$$= \frac{\sin(2x) + i\sinh 2y}{\cosh 2y - \cos 2x\ \cancel{\sin(x+iy)\sin(x-iy)}}$$

$$= \frac{\sin(2x)}{\cancel{\sin(x+iy)\sin(x-iy)}\ \cosh 2y - \cos 2x} + \frac{i\sinh 2y}{\cancel{\sin(x+iy)\sin(x-iy)}\ \cosh 2y - \cos 2x}$$

$$\text{Real part} = \frac{\sin(2x)}{\cosh 2y - \cos 2x}$$

$$\text{Imaginary part} = \frac{\sinh 2y}{\cosh 2y - \cos 2x}$$

16. Find the value of $\sinh^{-1}(3/4)$

**Soln:**

$\sinh^{-1}(3/4)$

W.K.T: $\sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right)$

put $x = 3/4$

$$\sinh^{-1}(3/4) = \log\left(3/4 + \sqrt{(3/4)^2+1}\right.$$

$= \log\left(\frac{3}{4} + \sqrt{\frac{9}{16}+1}\right)$

$= \log\left(\frac{3}{4} + \sqrt{\frac{25}{16}}\right)$

$= \log\left(\frac{3}{4} + \frac{5}{4}\right)$

$= \log\left(\frac{8}{4}\right)$

$\sinh^{-1}(3/4) = \log 2$

17. P.T. $\cosh^{-1}\left(\frac{x}{a}\right) = \log\left[\frac{x+\sqrt{x^2-a^2}}{a}\right]$

Soln:

W.K.T: $\cosh^{-1} x = \log\left[x + \sqrt{x^2-1}\right]$

Put $x = \frac{x}{a}$

$\cosh^{-1}\left(\frac{x}{a}\right) = \log\left[\frac{x}{a} + \sqrt{\frac{x^2}{a^2}-1}\right]$

$= \log\left[\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right]$

$= \log\left[\frac{x+\sqrt{x^2-a^2}}{a}\right]$

18. P.T: $\sinh 3x = 3\sinh x + 4\sin^3 h\, x$

Soln:

W.K.T: $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

Put $\theta = ix$

$\sin 3(ix) = 3\sin(ix) - 4\sin^3(ix)$

$i\sinh 3x = 3i\sinh x - 4i^3 \sin^3 hx$

$i\sinh 3x = 3i\sinh x + 4i \sin^3 hx$

$i(\sinh 3x) = i(3\sinh x + 4\sin^3 hx)$

$\sinh 3x = 3\sinh x + 4\sin^3 hx$

19. Separate real & imaginary part of $\tan^{-1}(\alpha+i\beta)$

Soln:

$$x+iy = \tan^{-1}(\alpha+i\beta)$$

$$\tan(x+iy) = \alpha+i\beta \cdot$$

Let $A = x+iy$

$B = x-iy$

$$2x = (x+iy)+(x-iy)$$

$$2iy = (x+iy)-(x-iy)$$

Take

$$\tan(2x) = \tan[(x+iy)+(x-iy)]$$

$$\tan 2x = \tan(x+iy)+\tan(x-iy)$$

$$\tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B}$$

$$\tan(2x) = \frac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy)\tan(x-iy)}$$

$$\tan 2x = \frac{(\alpha+i\beta)+(\alpha-i\beta)}{1-[(\alpha+i\beta)(\alpha-i\beta)]}$$

$$= \frac{2\alpha}{1-(\alpha^2-(i\beta)^2)}$$

$$\tan 2x = \frac{2\alpha}{1-(\alpha^2+\beta^2)}$$

$$2x = \tan^{-1}\left(\frac{2\alpha}{1-(\alpha^2+\beta^2)}\right)$$

$$\boxed{x = \frac{1}{2}\tan^{-1}\left(\frac{2x}{1-(\alpha^2+\beta^2)}\right)}$$

$2iy = (x+iy) - (x-iy)$

$\tan(2iy) = \tan[(x+iy) - (x-iy)]$

$$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B}$$

$$\tan(2iy) = \frac{\tan(x+iy) - \tan(x-iy)}{1+[\tan(x+iy)\tan(x-iy)]}$$

$$= \frac{(\alpha+i\beta) - (\alpha - i\beta)}{1+[(\alpha+i\beta)(\alpha-i\beta)]}$$

$$\tan 2iy = \frac{2i\beta}{1+(\alpha^2+\beta^2)}$$

$$2iy = \tan^{-1}\left(\frac{2i\beta}{1+(\alpha^2+\beta^2)}\right)$$

$$\boxed{y = \frac{1}{2}\tan^{-1}\left(\frac{2\beta}{1+(\alpha^2+\beta^2)}\right)}$$

20. Seperate into real & imaginary parts:

i) $\sin^{-1}(\cos\theta + i\sin\theta)$

let $x+iy = \sin^{-1}(\cos\theta + i\sin\theta)$

$\sin(x+iy) = \cos\theta + i\sin\theta$

$\sin x\cos(iy) + \cos x\sin(iy) = \cos\theta + i\sin\theta$

$\sin x\cosh y + i\cos x\sinh y = \cos\theta + i\sin\theta$

Equating the real & imaginary terms.

$\cos\theta = \sin x\cosh y \rightarrow (1)$

$\sin\theta = \cos x\sinh y$

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Squaring & adding ① & ②

$$\cos^2\theta + \sin^2\theta = \sin^2 x \cos^2 hy + \cos^2 x \sin^2 hy$$

$$1 = \sin^2 x (1 + \sin^2 hy) + (1 - \sin^2 x)\sin^2 hy$$

$$1 = \sin^2 x + \sin^2 x \sin^2 hy + \sin^2 hy - \sin^2 x \sin^2 hy$$

$$1 = \sin^2 x + \sin^2 hy$$

$$1 - \sin^2 x = \sin^2 hy$$

$$\cos x = \sin hy \rightarrow ③$$

③ in ②

$$\sin\theta = \cos x \cdot \cos x$$

$$\sin\theta = \cos^2 x$$

$$\sin\theta = (\cos x)^2$$

$$\cos x = \sqrt{\sin\theta}$$

$$x = \cos^{-1}\sqrt{\sin\theta}$$

$$\sin hy = \cos x$$

$$= \sqrt{\sin\theta}$$

$$\sin hy = \sqrt{\sin\theta}$$

$$y = \sin h^{-1}\left[\sqrt{\sin\theta}\right]$$

Real part $x = \cos^{-1}(\sqrt{\sin\theta})$

Imaginary part $y = \sin h^{-1}(\sqrt{\sin\theta})$.

ii) $\sin h(\alpha+i\beta)$

Soln:

$$x+iy = \sin h(\alpha+i\beta)$$

$$= \frac{1}{i}\sin(i(\alpha+i\beta))$$

$$= \frac{1}{i}\sin(i\alpha-\beta)$$

$$= -i[\sin(i\alpha)\cos\beta - \cos(i\alpha)\sin\beta]$$

$$= -i[i\sin h\alpha\cos\beta - \cos h\alpha\sin\beta]$$

$$x+iy = \sin h\alpha\cos\beta + i\cos h\alpha\sin\beta.$$

Real part $x = \sin h\alpha\cos\beta$

Imaginary part $y = \cos h\alpha\sin\beta$.

iii) $\tan h(1+i)$

Soln:

Let $x+iy = \tan h(1+i)$

$$= \frac{\sin h(1+i)}{\cos h(1+i)}$$

$$= \frac{1}{i}\frac{\sin(i(1+i))}{\cos(i(1+i))}$$

$$= -i\left[\frac{\sin(i-1)}{\cos(i-1)}\right]$$

$$= -i\left[\frac{\sin(i-1)}{\cos(i-1)} \times \frac{\cos(i+1)}{\cos(i+1)}\right]$$

$$= -i\left[\frac{2\sin(i-1)\cos(i+1)}{2\cos(i-1)\cos(i+1)}\right]$$

$$= -i\left[\frac{\sin(i-1+i+1) + \sin(i-1-i-1)}{\cos(i-1+i+1) + \cos(i-1-i-1)}\right]$$

$\frac{\sin h2}{\cos h2 - \cos 2}$

$\frac{\sin 2}{\cos h2 - \cos 2}$

$$= -i\left[\frac{\sin 2i + \sin(-2)}{\cos 2i + \cos(-2)}\right]$$

$$= -i\left[\frac{i \sinh 2 - \sin 2}{\cosh 2 + \cos 2}\right]$$

$$x + iy = \frac{\sinh 2 + i \sin 2}{\cosh 2 + \cos 2}$$

Real part $x = \dfrac{\sinh 2}{\cosh 2 + \cos 2}$

Imaginary part $y = \dfrac{\sin 2}{\cosh 2 + \cos 2}$

1. If $\cos\alpha + i\sin\alpha = \cos(\theta + i\phi)$

P.T $\sin^2\theta = \pm\sin\alpha$

Soln:

$\cos\alpha + i\sin\alpha = \cos(\theta + i\phi)$

$\cos\alpha + i\sin\alpha = \cos\theta\cos(i\phi) - \sin\theta\sin(i\phi)$

$\cos\alpha + i\sin\alpha = \cos\theta\cosh\phi - i\sin\theta\sinh\phi$

Equating real & imaginary terms.

$\cos\alpha = \cos\theta\cosh\phi$

$\frac{\cos\alpha}{\cos\theta} = \cosh\phi \rightarrow (1)$

$\sin\alpha = \sin\theta\sinh\phi$

$\frac{\sin\alpha}{\sin\theta} = \sinh\phi \rightarrow (2)$

Now squaring & subtract.

$\frac{\cos^2\alpha}{\cos^2\theta} - \frac{\sin^2\alpha}{\sin^2\theta} = \cosh^2\phi - \sinh^2\phi$

$\frac{\cos^2\alpha\sin^2\theta - \sin^2\alpha\cos^2\theta}{\cos^2\theta\sin^2\theta} = 1$

$(1-\sin^2\alpha)\sin^2\theta - \sin^2\alpha(1-\sin^2\theta) = (1-\sin^2\theta)\sin^2\theta$

$\sin^2\theta - \sin^2\alpha\sin^2\theta - \sin^2\alpha + \sin^2\alpha\sin^2\theta$

$= \sin^2\theta - \sin^2\theta\sin^2\theta$

$\sin^2\theta - \sin^2\alpha \sin^2\theta - \sin^2\alpha + \sin^2\alpha \sin^2\theta = \sin^2\theta - \sin^4\theta$

$\sin^2\theta - \sin^2\alpha = \sin^2\theta - \sin^4\theta$

$\sin^2\theta - \sin^2\alpha - \sin^2\theta + \sin^4\theta = 0$

$\sin^4\theta - \sin^2\alpha = 0$

$\sin^4\theta = \sin^2\alpha$

Taking square roots on both sides

$\sin^2\theta = \pm \sin\alpha$

2. If $(x+iy) = \tan(A+iB)$. P.T $x^2+y^2+2x\cot 2A = 1$

Soln:

$x+iy = \tan(A+iB)$

$x-iy = \tan(A-iB)$

$\cot 2A = \frac{1}{\tan 2A}$

$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A \tan B}$

$\cot 2A = \frac{1-\tan A \tan B}{\tan A + \tan B}$

$= \frac{1-\tan(A+iB)\tan(A-iB)}{\tan(A+iB)+\tan(A-iB)}$

$= \frac{1-(x+iy)(x-iy)}{x+iy+x-iy}$

$= \frac{1-(x^2-xiy+xiy+y^2)}{2x}$

$\cot 2A = \frac{1-x^2-y^2}{2x}$

$2x\cot 2A = 1-x^2-y^2$

$x^2+y^2+2x\cot 2A = 1$

Q. iv) $\tan^{-1}(x+iy)$.

Soln:

Let $x+iy = \tan^{-1}(x+iy)$

$\tan(x+iy) = x+iy$

$A = x+iy$

$B = x-iy$

$2x = (x+iy)+(x-iy)$

$2iy = (x+iy)-(x-iy)$

$\tan(2x) = \tan[(x+iy)+(x-iy)]$

$\tan(2x) = \tan(x+iy)+\tan(x-iy)$

$$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A \tan B}$$

$$\tan(2x) = \frac{\tan(x+iy)+\tan(x-iy)}{1-[\tan(x+iy)\tan(x-iy)]}$$

$$\tan 2x = \frac{x+iy+x-iy}{1-[(x+iy)(x-iy)]}$$

$$\tan 2x = \frac{2x}{1-(x^2+y^2)}$$

$$2x = \tan^{-1}\left(\frac{2x}{1-(x^2+y^2)}\right)$$

$$\boxed{x = \frac{1}{2}\tan^{-1}\left(\frac{2x}{1-(x^2+y^2)}\right)}$$

$2iy = (x+iy)-(x-iy)$

$\tan(2iy) = \tan[(x+iy)-(x-iy)]$

$\tan(2iy) = \tan(x+iy)-\tan(x-iy)$.

$$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B}$$

$$\tan(2iy) = \frac{\tan(x+iy) - \tan(x-iy)}{1+[\tan(x+iy)\cdot\tan(x-iy)]}$$

$$= \frac{x+iy-x+iy}{1+[(x+iy)(x-iy)]}$$

$$\tan 2iy = \frac{2iy}{1+(x^2+y^2)}$$

$$2iy = \tan^{-1}\left[\frac{2iy}{1+(x^2+y^2)}\right]$$

$$\boxed{y = \frac{1}{2}\tan^{-1}\left(\frac{2y}{1+(x^2+y^2)}\right)}$$

3. P.T $u = \log\tan\left[\frac{\pi}{4}+\frac{\theta}{2}\right]$ if $\cosh u = \sec\theta$

Soln:

$$\cosh u = \sec\theta$$

$$u = \cosh^{-1}(\sec\theta)$$

W.K.T: $\cosh^{-1}(x) = \log(x+\sqrt{x^2-1})$

$$\therefore u = \log(\sec\theta + \sqrt{\sec^2\theta - 1}$$

$$u = \log\left[\frac{1}{\cos\theta} + \sqrt{\tan^2\theta}\right]$$

$$u = \log\left[\frac{1}{\cos\theta} + \tan\theta\right]$$

$$u = \log\left[\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right]$$

Then put $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

$$\sin\theta = \frac{2\tan\theta/2}{1+\tan^2\theta/2}$$

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Put $t = \tan \theta/2$

$$\sin\theta = \frac{2t}{1+t^2}$$

$$\cos\theta = \frac{1-t^2}{1+t^2}$$

$$\therefore u = \log\left[\frac{1}{\frac{1-t^2}{1+t^2}} + \frac{2t/1+t^2}{\frac{1-t^2}{1+t^2}}\right]$$

$$= \log\left[\frac{1+t^2}{1-t^2} + \frac{2t}{1-t^2}\right]$$

$$= \log\left[\frac{1+t^2+2t}{1-t^2}\right]$$

$$= \log\left[\frac{(t+1)^2}{(1-t)(1+t)}\right]$$

$$u = \log\left[\frac{1+t}{1-t}\right]$$

$$= \log\left[\frac{\tan \pi/4 + \tan\theta/2}{1-\tan\pi/4 \tan\theta/2}\right]$$ $1 = \tan \pi/4$

$t = 1 \cdot t$

$= \tan\frac{\pi}{4} \tan\frac{\theta}{2}$

$$u = \log\left[\tan\left(\frac{\pi}{4} + \theta/2\right)\right]$$

4. If $\cos(x+iy) = r(\cos\alpha + i\sin\alpha)$.

P.T. $y = \frac{1}{2}\log\left[\frac{\sin(x-\alpha)}{\sin(x+\alpha)}\right]$

Soln :

$\cos(x+iy) = r\cos\alpha + ir\sin\alpha$

$\cos x \cos(iy) - \sin x \sin(iy) = r\cos\alpha + ir\sin\alpha$

$\cos x \cosh y - i\sin x \sinh y = r\cos\alpha + ir.\sin\alpha$

Equating Real & Imaginary terms.

$\cos x \cosh y = r\cos\alpha \rightarrow (1)$

$-\sin x \sinh y = r\sin\alpha \rightarrow (2)$

② ÷ ①

$\tan\alpha = -\tan x \tanh y$

$\tanh y = -\dfrac{\tan\alpha}{\tan x}$

$y = \tanh^{-1}\left[-\dfrac{\tan\alpha}{\tan x}\right]$

W.K.T: $\tanh^{-1}(x) = \dfrac{1}{2}\log\left[\dfrac{1+x}{1-x}\right]$

$\therefore x = -\dfrac{\tan\alpha}{\tan x}$

$\therefore y = \dfrac{1}{2}\log\left[\dfrac{1-\tan\alpha/\tan x}{1+\tan\alpha/\tan x}\right]$

$y = \dfrac{1}{2}\log\left[\dfrac{\dfrac{\tan x-\tan\alpha}{\tan x}}{\dfrac{\tan x+\tan\alpha}{\tan x}}\right]$

$y = \dfrac{1}{2}\log\left[\dfrac{\tan x-\tan\alpha}{\tan x+\tan\alpha}\right]$

$= \dfrac{1}{2}\log\left[\dfrac{\dfrac{\sin x}{\cos x}-\dfrac{\sin\alpha}{\cos\alpha}}{\dfrac{\sin x}{\cos x}+\dfrac{\sin\alpha}{\cos\alpha}}\right]$

$= \dfrac{1}{2}\log\left[\dfrac{\dfrac{\sin x\cos\alpha-\sin\alpha\cos x}{\cos x\cos\alpha}}{\dfrac{\sin x\cos\alpha+\sin\alpha\cos x}{\cos x\cos\alpha}}\right]$

$= \dfrac{1}{2}\log\left[\dfrac{\sin x\cos\alpha-\sin\alpha\cos x}{\sin x\cos\alpha+\sin\alpha\cos x}\right]$

$= \dfrac{1}{2}\log\left[\dfrac{\sin(x-\alpha)}{\sin(x+\alpha)}\right].$

5. P.T $\frac{1}{2}[\sin\theta + \sin h\theta] = \frac{\theta}{1!} + \frac{\theta^5}{5!} + \frac{\theta^7}{9!} + \dots + \infty$

Soln:

W.K.T: $\sin\theta = \frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} + \dots + \infty$

$\sin h\theta = \frac{\theta}{1!} + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \frac{\theta^7}{7!} + \frac{\theta^9}{9!} + \dots + \infty$

L.H.S:

$\frac{1}{2}[\sin\theta + \sin h\theta] = \frac{1}{2}\left[\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \frac{\theta^7}{7!} + \frac{\theta^9}{9!} + \dots + \infty\right] +$

$\left[\frac{\theta}{1!} + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \frac{\theta^7}{7!} + \frac{\theta^9}{9!} + \dots + \infty\right]$

$= \frac{1}{2}\left[\frac{2\theta}{1!} + \frac{2\theta^5}{5!} + \frac{2\theta^9}{9!} + \frac{2\theta^7}{7!} + \dots + \infty\right]$

$\frac{1}{2}[\sin\theta + \sin h\theta] = \frac{\theta}{1!} + \frac{\theta^5}{5!} + \frac{\theta^7}{7!} + \frac{\theta^9}{9!} + \dots + \infty$

6. P.T: $\frac{1}{2}[\cos\theta + \cos h\theta] = 1 + \frac{\theta^4}{4!} + \frac{\theta^8}{8!} + \dots + \infty$

Soln:

W.K.T: $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} + \dots + \infty$

$\cos h\theta = 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \frac{\theta^8}{8!} + \dots + \infty$

L.H.S.

$\frac{1}{2}[\cos\theta + \cos h\theta] = \frac{1}{2}\left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} + \dots + \infty\right] +$

$\left[1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \frac{\theta^8}{8!} + \dots + \infty\right]$

$= \frac{1}{2}\left[2 + \frac{2\theta^4}{4!} + \frac{2\theta^8}{8!} + \dots + \infty\right]$

$\frac{1}{2}[\cos\theta + \cos h\theta] = 1 + \frac{\theta^4}{4!} + \frac{\theta^8}{8!} + \dots + \infty$

7. If $x+iy = \cos(u+iv)$ where $x, y, u, v$ are real

P.T:

i) $(1+x)^2 + y^2 = (\cosh v + \cos u)^2$

ii) $(1-x)^2 + y^2 = (\cosh v - \cos u)^2$

Soln

Let $x+iy = \cos(u+iv)$

$x+iy = \cos u \cos(iv) - \sin u \sin(iv)$

$= \cos u \cosh v - i \sin u \sinh v$.

Equating real & imaginary parts

$x = \cos u \cosh v \Rightarrow x^2 = \cos^2 u \cosh^2 v$

$y = -\sin u \sinh v \Rightarrow y^2 = \sin^2 u \sinh^2 v$

i) $(1+x)^2 + y^2 = (1+\cos u \cosh v)^2 + (\sin^2 u \sinh^2 v)$

$= 1 + \cos^2 u \cosh^2 v + 2\cos u \cosh v + \sin^2 u \sinh^2 v$

$= 1 + \cos^2 u \cosh^2 v + 2\cos u \cosh v + (1-\cos^2 u)(\cosh^2 v - 1)$

$= 1 + \cos^2 u \cosh^2 v + 2\cos u \cosh v + \cosh^2 v - 1 - \cos^2 u \cosh^2 v + \cos^2 u$

$= \cosh^2 v + \cos^2 u + 2\cos u \cosh v$

$(1+x)^2 + y^2 = (\cosh v + \cos u)^2$

ii) $(1-x)^2 + y^2 = (1-\cos u \cosh v)^2 + (\sin^2 u \sinh^2 v)$

$= 1 + \cos^2 u \cosh^2 v - 2\cos u \cosh v + \sin^2 u \sinh^2 v$

$= 1 + \cos^2 u \cosh^2 v - 2\cos u \cosh v + (1-\cos^2 u)(\cosh^2 v - 1)$

$= 1 + \cos^2 u \cosh^2 v - 2\cos u \cosh v + \cosh^2 v - 1 - \cos^2 u \cosh^2 v + \cos^2 u$

$= \cosh^2 v + \cos^2 u - 2\cos u \cosh v$

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$$(1-x)^2+y^2=(\cos h v-\cos u)^2$$

8. $\cot h^{-1}(x)=\frac{1}{2}\log\left[\frac{x+1}{x-1}\right]$

Soln :

Put $y=\cot h^{-1}(x)$

$$x=\cot h(y)$$

$$x=\frac{\cos hy}{\sin hy}$$

$$x=\frac{\frac{e^y+e^{-y}}{2}}{\frac{e^y-e^{-y}}{2}}$$

$$x=\frac{e^y+e^{-y}}{e^y-e^{-y}}$$

$$x(e^y-e^{-y})=e^y+e^{-y}$$

$$xe^y-xe^{-y}=e^y+e^{-y}$$

$$xe^y-e^y=e^{-y}+xe^{-y}$$

$$e^y(x-1)=e^{-y}(x+1)$$

$$\frac{e^y}{e^{-y}}(x-1)=x+1$$

$$e^y.e^y=\frac{x+1}{x-1}$$

$$e^{2y}=\frac{x+1}{x-1}$$

Taking log on both sides

$$2y=\log\left[\frac{x+1}{x-1}\right]$$

$$y=\frac{1}{2}\log\left[\frac{x+1}{x-1}\right]$$

$$\cot h^{-1}(x)=\frac{1}{2}\log\left[\frac{x+1}{x-1}\right]$$

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9. P.T $\coth^{-1}\left[\frac{x^2+1}{x^2-1}\right] = \log x$

<u>soln</u>:

W.K.T: $\cot h^{-1}(x) = \frac{1}{2}\log\left[\frac{x+1}{x-1}\right]$

put $x = \frac{x^2+1}{x^2-1}$

$$\cot h^{-1}\left[\frac{x^2+1}{x^2-1}\right] = \frac{1}{2}\log\left[\frac{\left[\frac{x^2+1}{x^2-1}\right]+1}{\left[\frac{x^2+1}{x^2-1}\right]-1}\right]$$

$$= \frac{1}{2}\log\left[\frac{\frac{x^2+1+x^2-1}{x^2-1}}{\frac{x^2+1-x^2+1}{x^2-1}}\right]$$

$$= \frac{1}{2}\log\left[\frac{2x^2}{2}\right]$$

$$= \frac{1}{2}\log x^2$$

$$= \frac{2}{2}\log x$$

$$\cot h^{-1}\left[\frac{x^2+1}{x^2-1}\right] = \log x.$$

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