UNIT - III

Expression for $\sin n\theta$, $\cos n\theta$ and $\tan n\theta$.

Theorem:

For any positive integer n

i) $\cos n\theta = \cos^n\theta - nC_2 \cos^{n-2}\theta \sin^2\theta + \dots$

ii) $\sin n\theta = nC_1 \cos^{n-1}\theta \sin\theta - nC_3 \cos^{n-3}\theta \sin^3\theta + \dots$

Proof:

By De-moivre's theorem,

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$$

$$= \cos^n\theta + nC_1 \cos^{n-1}\theta (i\sin\theta) + nC_2 \cos^{n-2}\theta (i\sin\theta)^2 + nC_3 \cos^{n-3}\theta (i\sin\theta)^3 + nC_4 \cos^{n-4}\theta (i\sin\theta)^4 + \dots$$

$$\cos n\theta + i\sin n\theta = \cos^n\theta + i\,nC_1 \cos^{n-1}\theta \sin\theta - nC_2 \cos^{n-2}\theta \sin^2\theta - i\,nC_3 \cos^{n-3}\theta \sin^3\theta + nC_4 \cos^{n-4}\theta \sin^4\theta + \dots$$

Equating real and imaginary parts

i) $\cos n\theta = \cos^n\theta - nC_2 \cos^{n-2}\theta \sin^2\theta + \dots$

ii) $\sin n\theta = nC_1 \cos^{n-1}\theta \sin\theta - nC_3 \cos^{n-3}\theta \sin^3\theta + \dots$

Corollary:

Prove that: $\tan n\theta = \dfrac{nC_1 \tan\theta - nC_3 \tan^3\theta + \dots}{1 - nC_2 \tan^2\theta + nC_4 \tan^4\theta - \dots}$

Proof:

$$\tan n\theta = \frac{\sin n\theta}{\cos n\theta}$$

$$= \frac{nC_1 \cos^{n-1}\theta \sin\theta - nC_3 \cos^{n-3}\theta \sin^3\theta + \dots}{\cos^n\theta - nC_2 \cos^{n-2}\theta \sin^2\theta + nC_4 \cos^{n-4}\theta \sin^4\theta - \dots}$$

$$= \frac{nC_1 \cos^n\theta \dfrac{\sin\theta}{\cos\theta} - nC_3 \cos n\theta \dfrac{\sin^3\theta}{\cos^3\theta} + \dots}{\cos^n\theta - nC_2 \cos^n\theta \dfrac{\sin^2\theta}{\cos^2\theta} + nC_4 \cos^n\theta \dfrac{\sin^4\theta}{\cos^4\theta} - \dots}$$

$$= \frac{\cos^n\theta\,[nC_1 \tan\theta - nC_3 \tan^3\theta + \dots]}{\cos^n\theta\,[1 - nC_2 \tan^2\theta + nC_4 \tan^4\theta - \dots]}$$

$$\tan n\theta = \frac{nC_1 \tan\theta - nC_3 \tan^3\theta + \dots}{1 - nC_2 \tan^2\theta + nC_4 \tan^4\theta - \dots}$$

Theorem:

P.T: $\tan(A_1 + A_2 + A_3 + \dots + A_n) = \dfrac{S_1 - S_3 + S_5 - \dots}{1 - S_2 + S_4 - \dots}$

Where $S_1 = \Sigma \tan A_i$; $S_2 = \Sigma \tan A_i \tan A_j$; ... ;

$S_n = \tan A_1 \tan A_2 \dots \tan A_n$.

Proof:

Consider,

$\cos A_1 + i\sin A_1 = \cos A_1 \left(1 + \dfrac{i\sin A_1}{\cos A_1}\right)$

$\cos A_1 + i\sin A_1 = \cos A_1 (1 + i\tan A_1)$

$\cos A_2 + i\sin A_2 = \cos A_2 (1 + i\tan A_2)$

$\cos A_3 + i\sin A_3 = \cos A_3 (1 + i\tan A_3)$

⋮

$\cos A_n + i\sin A_n = \cos A_n (1 + i\tan A_n)$

Multiply these equations.

$\cos(A_1 + A_2 + A_3 + \dots + A_n) + i\sin(A_1 + A_2 + A_3 + \dots + A_n)$

$= \cos A_1 \cos A_2 \dots \cos A_n (1 + iS_1 + i^2 S_2 + i^3 S_3 + \dots)$

$\cos(A_1+A_2+A_3+\cdots+A_n)+i\sin(A_1+A_2+A_3+\cdots+A_n)$

$= \cos A_1 \cos A_2 \cdots \cos A_n(1+iS_1-S_2-iS_3+\cdots)$

Equating real & imaginary parts.

$\cos(A_1+A_2+A_3+\cdots+A_n) = \cos A_1 \cos A_2 \cdots \cos A_n (1-S_2+\cdots) \rightarrow (1)$

$\sin(A_1+A_2+A_3+\cdots+A_n) = \cos A_1 \cos A_2 \cdots \cos A_n (S_1-S_3+S_5-\cdots) \rightarrow (2)$

(2) $\div$ (1) we get.

$$\frac{\sin(A_1+A_2+A_3+\cdots+A_n)}{\cos(A_1+A_2+A_3+\cdots+A_n)} = \frac{\cos A_1 \cos A_2 \cdots \cos A_n (S_1-S_3+S_5-\cdots)}{\cos A_1 \cos A_2 \cdots \cos A_n (1-S_2+\cdots)}$$

$$\tan(A_1+A_2+A_3+\cdots+A_n) = \frac{S_1-S_3+S_5-\cdots}{1-S_2+\cdots}$$

1. Expand $\sin 7\theta$ in power of $\cos\theta$ and $\sin\theta$. Hence

P.T: $\dfrac{\sin 7\theta}{\cos\theta} = 7-56\sin^2\theta+112\sin^4\theta-64\sin^6\theta$

Soln:

$\cos 7\theta + i\sin 7\theta = (\cos\theta + i\sin\theta)^7$

$= \cos^7\theta + 7C_1\cos^6\theta(i\sin\theta) + 7C_2\cos^5\theta$
$(i\sin\theta)^2 + 7C_3\cos^4\theta(i\sin\theta)^3 +$
$7C_4\cos^3\theta(i\sin\theta)^4 + 7C_5\cos^2\theta$
$(i\sin\theta)^5 + 7C_6\cos\theta(i\sin\theta)^6 + 7C_7(i\sin\theta)^7$

$\cos 7\theta + i\sin 7\theta = \cos^7\theta + i7\cos^6\theta\sin\theta - 21\cos^5\theta\sin^2\theta$
$- i35\cos^4\theta\sin^3\theta + 35\cos^3\theta\sin^4\theta$
$+ i21\cos^2\theta\sin^5\theta - 7\cos\theta\sin^6\theta -$
$i\sin^7\theta$

$a^3-3a^2b+3ab^2-b^3$

Equating Imaginary parts

$\sin 7\theta = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta$
$- \sin^7\theta$

$\frac{\sin 7\theta}{\sin\theta} = 7\cos^6\theta - 35\cos^4\theta\sin^2\theta + 21\cos^2\theta\sin^4\theta - \sin^6\theta$

$= 7(\cos^2\theta)^3 - 35(\cos^2)^2\sin^2\theta + 21\cos^2\theta\sin^4\theta - \sin^6\theta$

$= 7(1-\sin^2\theta)^3 - 35(1-\sin^2\theta)^2\sin^2\theta + 21\,\cancel{\cos^2\theta}\,(1-\sin^2\theta)\sin^4\theta - \sin^6\theta$

$= 7(1-3\sin^2\theta+3\sin^4\theta-\sin^6\theta) - 35(1+\sin^4\theta-2\sin^2\theta)\sin^2\theta + 21(\sin^4\theta-\sin^6\theta) - \sin^6\theta$

$= 7 - 21\sin^2\theta + 21\sin^4\theta - 7\sin^6\theta - (35\sin^2\theta + 35\sin^4\theta - 70\sin^2\theta)\sin^2\theta + 21\sin^4\theta - 21\sin^6\theta - \sin^6\theta$

$= 7 - 21\sin^2\theta + 21\sin^4\theta - 7\sin^6\theta - 35\sin^2\theta - 35\sin^6\theta + 70\sin^4\theta + 21\sin^4\theta - 21\sin^6\theta - \sin^6\theta$

$\frac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$

2. P.T: $\cos 8\theta = 128\cos^8\theta - 256\cos^6\theta + 160\cos^4\theta - 32\cos^2\theta + 1$

Soln:

$(\cos 8\theta + i\sin 8\theta) = (\cos\theta + i\sin\theta)^8$

$= \cos^8\theta + 8C_1\cos^7\theta(i\sin\theta) + 8C_2\cos^6\theta(i\sin\theta)^2 + 8C_3\cos^5\theta(i\sin\theta)^3 + 8C_4\cos^4\theta(i\sin\theta)^4 + 8C_5\cos^3\theta(i\sin\theta)^5 + 8C_6\cos^2\theta(i\sin\theta)^6 + 8C_7\cos\theta(i\sin\theta)^7 + 8C_8(i\sin\theta)^8$

$\cos 8\theta + i\sin 8\theta = \cos^8\theta + i\,8\cos^7\theta\sin\theta - 28\cos^6\theta\sin^2\theta$
$- i\,56\cos^5\theta\sin^3\theta + 70\cos^4\theta\sin^4\theta$
$- i\,56\cos^3\theta\sin^5\theta - 28\cos^2\theta\sin^6\theta$
$- i\,8\cos\theta\sin^7\theta + \sin^8\theta$

Equating real parts on both sides.

$\cos 8\theta = \cos^8\theta - 28\cos^6\theta\sin^2\theta + 70\cos^4\theta\sin^4\theta$
$- 28\cos^2\theta\sin^6\theta + \sin^8\theta$
$= \cos^8\theta - 28\cos^6\theta(1-\cos^2\theta) + 70\cos^4\theta$
$(1-\cos^2\theta)^2 - 28\cos^2\theta(1-\cos^2\theta)^3 + (1-\cos^2\theta)^4$
$= \cos^8\theta - 28\cos^6\theta + 28\cos^8\theta + 70\cos^4\theta$
$(1+\cos^4\theta - 2\cos^2\theta) - 28\cos^2\theta(1+\cos^6\theta -$
$8\cos^2\theta + 3\cos^4\theta) + (1+\cos^4\theta - 2\cos^2\theta)^2$
$= \cos^8\theta - 28\cos^6\theta + 28\cos^8\theta + 70\cos^4\theta +$
$70\cos^8\theta - 140\cos^6\theta - 28\cos^2\theta + 28\cos^8\theta$
$+ 84\cos^2\theta - 84\cos^6\theta + 1 + \cos^8\theta + 4\cos^4\theta$
$+ 2\cos^4\theta - 4\cos^6\theta - 4\cos^2\theta$

$\cos 8\theta = 128\cos^8\theta - 256\cos^6\theta + 160\cos^4\theta - 32\cos^2\theta + 1$

$(a+b+c)^2 = a^2+b^2+c^2+2ab-2bc-2ca$

3. P.T: $\tan 7\theta = \dfrac{7\tan\theta - 35\tan^3\theta + 21\tan^5\theta - \tan^7\theta}{1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta}$

Soln:

$\tan n\theta = \dfrac{nC_1\tan\theta - nC_3\tan^3\theta + nC_5\tan^5\theta - \dots}{1 - nC_2\tan^2\theta + nC_4\tan^4\theta - nC_6\tan^6\theta + \dots}$

$$= \frac{7C_1 \tan\theta - 7C_3 \tan^3\theta + 7C_5 \tan^5\theta - 7C_7 \tan^7\theta}{1 - 7C_2 \tan^2\theta + 7C_4 \tan^4\theta - 7C_6 \tan^6\theta}$$

$$\tan 7\theta = \frac{7\tan\theta - 35\tan^3\theta + 21\tan^5\theta - \tan^7\theta}{1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta}$$

4. Prove that the eqn, $a^2\cos^2\theta + b^2\sin^2\theta + 2ag\cos\theta + 2bf\sin\theta + c = 0$

~~Soln:~~ Has 4 routes and that the sum of the 4 values of θ is an even multiple of π.

Soln:

$a^2\cos^2\theta + b^2\sin^2\theta + 2ag\cos\theta + 2bf\sin\theta + c = 0$

Gin, $a^2\cos^2\theta + b^2\sin^2\theta + 2ag\cos\theta$

$$a^2\left(\frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}\right)^2 + b^2\left(\frac{2\tan\theta/2}{1+\tan^2\theta/2}\right)^2 + 2ag\left(\frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}\right)$$
$$+ 2bf\left(\frac{2\tan\theta/2}{1+\tan^2\theta/2}\right) + c = 0$$

Put $\tan\theta/2 = t$

$$a^2\left(\frac{1-t^2}{1+t^2}\right)^2 + b^2\left(\frac{2t}{1+t^2}\right)^2 + 2ag\left(\frac{1-t^2}{1+t^2}\right) + 2bf\left(\frac{2t}{1+t^2}\right) + c = 0$$

$$a^2\frac{(1-t^2)^2}{(1+t^2)^2} + b^2\frac{(4t^2)}{(1+t^2)^2} + 2ag\frac{(1-t^2)}{(1+t^2)} + 2bf\frac{(2t)}{(1+t^2)} + c = 0$$

$$a^2(1+t^4-2t^2) + 4b^2t^2 + 2ag(1-t^2)(1+t^2) + 4bft(1+t^2)$$
$$+ c(1+t^2)^2 = 0$$

$$a^2 + a^2t^4 - 2a^2t^2 + 4b^2t^2 + 2ag(1-t^4) + 4bft + 4bft^3$$
$$+ c(1+t^4+2t^2) = 0$$

$$a^2 + a^2t^4 - 2a^2t^2 + 4b^2t^2 + 2ag - 2agt^4 + 4bft + 4bft^3$$
$$+ c + ct^4 + 2ct^2 = 0$$

$$t^4(a^2 - 2ag + c) + t^3(4bf) + t^2(-2a^2 + 4b^2 + 2c) + t(4bf)$$
$$+ (a^2 + 2ag + c) = 0$$

$$t^4 + \left(\frac{4af}{a^2-2ag+c}\right)t^3 + \left(\frac{-2a^2+4b^2+2c}{a^2-2ag+c}\right)t^2 +$$

$$+ \left(\frac{4bf}{a^2-2ag+c}\right) + \left(\frac{a^2+2ag+c}{a^2-2ag+c}\right) = 0$$

This is a fourth ~~square~~ degree eqn. in t and has 4 roots say $t_i = \tan(\theta_i/2)$, $i = 1,2,3,4$.

Now $\tan\left(\frac{\theta_1}{2} + \frac{\theta_2}{2} + \frac{\theta_3}{2} + \frac{\theta_4}{2}\right) = \frac{S_1 - S_3}{1 - S_2 + S_4}$

$$S_1 = \Sigma \tan\left(\frac{\theta_1}{2}\right) = \Sigma t_1 = \frac{-4bf}{a^2-2ag+c}$$

$$S_3 = \Sigma \tan\left(\frac{\theta_1}{2}\right)\tan\left(\frac{\theta_2}{2}\right)\tan\left(\frac{\theta_3}{2}\right) = \Sigma t_1 t_2 t_3 = \frac{-4bf}{a^2-2ag+c}$$

$$\therefore \tan\left(\frac{\theta_1}{2} + \frac{\theta_2}{2}, \frac{\theta_3}{2} + \frac{\theta_4}{2}\right) = 0$$

$$\therefore \frac{\theta_1}{2} + \frac{\theta_2}{2} + \frac{\theta_3}{2} + \frac{\theta_4}{2} = n\pi \text{ where } n \in Z$$

$$\cancel{\therefore \frac{\theta_1}{2} + \frac{\theta_2}{2} + \frac{\theta_3}{2} + \frac{\theta_4}{2} = 2n\pi}$$

$$\therefore \theta_1 + \theta_2 + \theta_3 + \theta_4 = 2n\pi \text{ where } n \in Z.$$

Expression for $\sin^n\theta$ and $\cos^n\theta$.

Theorem:

When n is a positive integer,

$$\cos^n\theta = \frac{1}{2^{n-1}}\left[\cos n\theta + nC_1\cos(n-2)\theta + nC_2\cos(n-4)\theta + \cdots\right]$$

Proof:

Let $x = \cos\theta + i\sin\theta$

$$\frac{1}{x} = \cos\theta - i\sin\theta$$

$$x + \frac{1}{x} = 2\cos\theta \rightarrow (1)$$

$$x^n + \frac{1}{x^n} = 2\cos n\theta.$$

$$2\cos\theta = x + \frac{1}{x}$$

$$(2\cos\theta)^n = \left(x + \frac{1}{x}\right)^n$$

$$2^n\cos^n\theta = x^n + nC_1 x^{n-1}\left(\frac{1}{x}\right) + nC_2 x^{n-2}\left(\frac{1}{x^2}\right) + \dots + nC_{n-1}\, x\left(\frac{1}{x^{n-1}}\right) + nC_n\left(\frac{1}{x^n}\right).$$

$$2^n\cos^n\theta = x^n + nC_1 x^{n-2} + nC_2 x^{n-4} + nC_{n-1}\left(\frac{1}{x^{n-2}}\right) + nC_n\left(\frac{1}{x^n}\right).$$

Since, $\because nC_{n-1} = nC_1,\ nC_{n-2} = nC_2 \dots$

$$2^n\cos^n\theta = x^n + nC_1 x^{n-2} + nC_2 x^{n-4} + \dots + nC_1\left(\frac{1}{x^{n-2}}\right) + \frac{1}{x^n}$$

$$= \left(x^n + \frac{1}{x^n}\right) + nC_1\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + nC_2\left(x^{n-4} + \frac{1}{x^{n-4}}\right) + \dots$$

$$= 2\cos n\theta + nC_1\left[2\cos(n-2)\theta\right] + nC_2\left[2\cos(n-4)\theta\right] + \dots$$

$$= 2\left[\cos n\theta + nC_1\cos(n-2)\theta + nC_2\cos(n-4)\theta + \dots\right]$$

$$\cos^n\theta = \frac{2}{2^n}\left[\cos n\theta + nC_1\cos(n-2)\theta + nC_2\cos(n-4)\theta + \dots\right]$$

$$\cos^n\theta = \frac{1}{2^{n-1}}\left[\cos n\theta + nC_1\cos(n-2)\theta + nC_2\cos(n-4)\theta + \dots\right]$$

Theorem:

When n is a positive integer,

$$\sin^n\theta = \begin{cases} \dfrac{1}{(-1)^{n/2}\, 2^{n-1}}\left[\cos n\theta - nC_1\cos(n-2)\theta + nC_2\cos(n-4)\theta - \dots + \dfrac{(-1)^{n/2}}{2} nC_{n/2}\right], & \text{if } n \text{ is even} \\[2ex] \dfrac{1}{(-1)^{\frac{n-1}{2}}\, 2^{n-1}}\left[\sin n\theta - nC_1\sin(n-2)\theta + nC_2\sin(n-4)\theta - \dots + (-1)^{\frac{n-1}{2}} nC_{\frac{n-1}{2}}\sin\theta\right], & \text{if } n \text{ is odd} \end{cases}$$

Proof:

Let $x = \cos\theta + i\sin\theta$

$\frac{1}{x} = \cos\theta - i\sin\theta$

$x - \frac{1}{x} = 2i\sin\theta$

$x^n - \frac{1}{x^n} = 2i\sin n\theta$.

$2i\sin n\theta = x - \frac{1}{x}$

$(2i\sin\theta)^n = (x - \frac{1}{x})^n$

$2^n i^n \sin^n\theta = x^n - nC_1 x^{n-1}(\frac{1}{x}) + nC_2 x^{n-2}(\frac{1}{x^2}) - \dots + (-1)^n nC_n (\frac{1}{x^n})$

$2^n i^n \sin^n\theta = x^n - nC_1 x^{n-2} + nC_2 x^{n-4} - nC_3 x^{n-6} + \dots + (-1)^n (\frac{1}{x^n}) \rightarrow (1)$

Case (i):

n is even.

$\therefore$ The number of terms in the right side is add and hence the last term is positive and the last but one term is negative etc.. and the middle term independent of x and Also the middle term independent of x and it is $(-1)^{n/2} nC_{n/2}$.

Also, $i^n = (i^2)^{n/2}$

$i^n = (-1)^{n/2}$ in eqn (1)

Using $nC_r = nC_{n-r}$

$2^n(-1)^{n/2}\sin^n\theta = (x^n + \frac{1}{x^n}) - nC_1(x^{n-2} + \frac{1}{x^{n-2}}) + nC_2 (x^{n-4} + \frac{1}{x^{n-4}}) \mp \dots + (-1)^{n/2} nC_{n/2}$

$= 2\cos n\theta - nC_1[2\cos(n-2)\theta] + nC_2 [2\cos(n-4)\theta] \mp \dots + (-1)^{n/2} nC_{n/2}$

$$2^n (-1)^{n/2} \sin^n \theta = 2[\cos n\theta - nC_1 \cos(n-2)\theta + nC_2 \cos(n-4)\theta \cdots (-1)^{n/2} nC_{n/2}]$$

$$\sin^n\theta = \frac{2}{2^n(-1)^{n/2}} [\cos n\theta - nC_1 \cos(n-2)\theta + nC_2 \cos(n-4)\theta - \cdots (-1)^{n/2} nC_{n/2}]$$

$$\sin^n\theta = \frac{1}{(-1)^{n/2} 2^{n-1}} [\cos n\theta - nC_1 \cos(n-2)\theta + nC_2 \cos(n-4)\theta \cdots (-1)^{n/2} nC_{n/2}]$$

Case (ii) : n is odd

The number of terms in the right side of eqn ① is even and hence the last term is negative and the last but one term is positive etc..

Also there are 2 middle terms are $(-1)^{\frac{n-1}{2}} nC_{\frac{n-1}{2}} x$ and $(-1)^{\frac{n+1}{2}} nC_{\frac{n+1}{2}} (\frac{1}{x})$

Using $nC_r = nC_{n-r}$ in eqn ①

$$(2i \sin\theta) = x - \frac{1}{x}$$

$$(2i \sin\theta)^n = (x - \frac{1}{x})^n$$

$$2^n i^n \sin^n\theta = (x^n - \frac{1}{x^n}) - nC_1 (x^{n-2} - \frac{1}{x^{n-2}}) + nC_2 (x^{n-4} - \frac{1}{x^{n-4}}) \cdots (-1)^{\frac{n-1}{2}} nC_{\frac{n-1}{2}} (x - \frac{1}{x})$$

$$= (2i \sin n\theta) - nC_1 [2i \sin(n-2)\theta] + nC_2 [2i \sin(n-4)\theta] \cdots (-1)^{\frac{n-1}{2}} nC_{\frac{n-1}{2}} (2i \sin\theta)$$

$$2^n i^n \sin^n\theta = 2i [\sin n\theta - nC_1 (\sin(n-2)\theta) + nC_2 (\sin(n-4)\theta) \cdots (-1)^{\frac{n-1}{2}} nC_{\frac{n-1}{2}} \sin\theta].$$

$$\sin^n\theta = \frac{2i}{2^n i^n} [\sin n\theta - nC_1 (\sin(n-2)\theta) + nC_2 (\sin(n-4)\theta)$$

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$$\cdots (-1)^{\frac{n-1}{2}}\, nC_{\frac{n-1}{2}} \sin\theta]$$

$$\sin^n\theta = \frac{1}{2^{n-1} i^{n-1}} [\sin n\theta - nC_1 (\sin(n-2)\theta) + nC_2 (\sin(n-4)\theta)$$
$$\cdots (-1)^{\frac{n-1}{2}}\, nC_{\frac{n-1}{2}} \sin\theta]$$

$$= \frac{1}{2^{n-1} (-1)^{\frac{n-1}{2}}} [\sin n\theta - nC_1 (\sin(n-2)\theta) + nC_2 (\sin(n-4)\theta)$$
$$\cdots (-1)^{\frac{n-1}{2}}\, nC_{\frac{n-1}{2}} \sin\theta].$$

5. P.T: $\sin 5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$

Soln:

$$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$$
$$= \cos^5\theta + 5C_1 \cos^4\theta\, (i\sin\theta) + 5C_2 \cos^3\theta\, (i\sin\theta)^2 + 5C_3 \cos^2\theta\, (i\sin\theta)^3 + 5C_4 \cos\theta\, (i\sin\theta)^4 + 5C_5 (i\sin\theta)^5$$
$$= \cos^5\theta + i5\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - i10\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$$

Equating the imaginary parts of both sides.

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$
$$= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$$
$$= 5(1+\sin^4\theta - 2\sin^2\theta)\sin\theta - 10(\sin^3\theta - \sin^5\theta) + \sin^5\theta$$
$$= (5+5\sin^4\theta - 10\sin^2\theta)\sin\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$$
$$= 5\sin\theta + 5\sin^5\theta - 10\sin^3\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$$

$$\sin 5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta.$$

6. $\frac{\cos 9\theta}{\cos\theta} = 256\cos^8\theta - 576\cos^6\theta + 432\cos^4\theta - 120\cos^2\theta + 9$

Soln:

$\cos 4\theta + i\sin 4\theta = (\cos\theta + i\sin\theta)^4$

$= \cos^4\theta + 4C_1\cos^3\theta(i\sin\theta) + 4C_2 \cos^2\theta(i\sin\theta)^2 + 4C_3\cos^3\theta(i\sin\theta)^3 + 4C_4(i\sin\theta)^4$

$= \cos^4\theta + i4\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - i4\cos^3\theta\sin^3\theta + \sin^4\theta$

$\cos 9\theta + i\sin 9\theta = (\cos\theta + i\sin\theta)^9$

$= \cos^9\theta + 9C_1\cos^8\theta(i\sin\theta) + 9C_2\cos^7\theta(i\sin\theta)^2 + 9C_3\cos^6\theta(i\sin\theta)^3 + 9C_4\cos^5\theta(i\sin\theta)^4 + 9C_5\cos^4\theta(i\sin\theta)^5 + 9C_6\cos^3\theta(i\sin\theta)^6 + 9C_7\cos^2\theta(i\sin\theta)^7 + 9C_8\cos\theta(i\sin\theta)^8 + 9C_9(i\sin\theta)^9$

$= \cos^9\theta + i9\cos^8\theta\sin\theta - 36\cos^7\theta\sin^2\theta - i84\cos^6\theta\sin^3\theta + 126\cos^5\theta\sin^4\theta + i126\cos^4\theta\sin^5\theta - 84\cos^3\theta\sin^6\theta - i36\cos^2\theta\sin^7\theta + 9\cos\theta\sin^8\theta + \sin^9\theta$

Equating real parts on both sides.

$\cos 9\theta = \cos^9\theta - 36\cos^7\theta\sin^2\theta + 126\cos^5\theta\sin^4\theta - 84\cos^3\theta\sin^6\theta + 9\cos\theta\sin^8\theta$

$= \cos 9\theta - 36\cos^7\theta(1-\cos^2\theta) + 126\cos^5\theta(1-\cos^2\theta)^2 - 84\cos^3\theta(1-\cos^2\theta)^3 + 9\cos(1-\cos^2\theta)^4.$

7. $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

Soln:

$$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$$

$$= \cos^5\theta + 5C_1 \cos^4\theta (i\sin\theta) + 5C_2 \cos^3\theta (i\sin\theta)^2 + 5C_3 \cos^2\theta (i\sin\theta)^3 + 5C_4 \cos\theta (i\sin\theta)^4 + 5C_5 (i\sin\theta)^5$$

$$\cos 5\theta + i\sin\theta = \cos^5\theta + i5\cos^4\theta \sin\theta - 10\cos^3\theta \sin^2\theta - i10\cos^2\theta \sin^3\theta + 5\cos\theta \sin^4\theta + i\sin^5\theta.$$

Equating real parts on both sides.

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta \sin^2\theta + 5\cos\theta \sin^4\theta$$

$$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$$

$$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta\,(1+\cos^4\theta - 2\cos^2\theta)$$

$$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta + 5\cos^5\theta$$

$$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta + 5\cos^5\theta - 10\cos^3\theta$$

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$

8. Find $\tan 8\theta$:

Soln:

$$\tan n\theta = \frac{nC_1 \tan\theta - nC_3 \tan^3\theta + nC_5 \tan^5\theta - \dots}{1 - nC_2 \tan^2\theta + nC_4 \tan^4\theta - nC_6 \tan^6\theta + \dots}$$

$$\tan 8\theta = \frac{8C_1 \tan\theta - 8C_3 \tan^3\theta + 8C_5 \tan^5\theta - 8C_7 \tan^7\theta}{1 - 8C_2 \tan^2\theta + 8C_4 \tan^4\theta - 8C_6 \tan^6\theta + 8C_8 \tan^8\theta}$$

$$\tan 8\theta = \frac{8\tan\theta - 56\tan^3\theta + 56\tan^5\theta - 8\tan^7\theta}{1 - 28\tan^2\theta + 70\tan^4\theta - 28\tan^6\theta + \tan^8\theta}.$$

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$\cos 4\theta + 15\cos 2\theta + 10$

1. P.T: $2^5\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$

Soln:

Let $x = \cos\theta + i\sin\theta$

$\frac{1}{x} = \cos\theta - i\sin\theta$

$x + \frac{1}{x} = 2\cos\theta \quad ; x^n + \frac{1}{x^n} = 2\cos n\theta$

$(x + \frac{1}{x})^6 = (2\cos\theta)^6$

$2^6\cos^6\theta = (x + \frac{1}{x})^6$

$2^n\cos^n\theta = (x^n + \frac{1}{x^n}) + nC_1(x^{n-2} + \frac{1}{x^{n-2}}) + nC_2(x^{n-4} + \frac{1}{x^{n-4}}) + \dots$

$2^6\cos^6\theta = (x^6 + \frac{1}{x^6}) + 6C_1(x^4 + \frac{1}{x^4}) + 6C_2(x^2 + \frac{1}{x^2}) + 6C_3 (x^0 + \frac{1}{x^0})$

$= 2\cos 6\theta + 6(2\cos 4\theta) + 15(2\cos 2\theta) + 20$

$2^6\cos^6\theta = 2[\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10]$

$2^5\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10.$

2. P.T: $\sin^5\theta = \frac{1}{2^4}[\sin 5\theta - 5\sin 3\theta + 10\sin\theta]$

Soln:

$x = \cos\theta + i\sin\theta$

$\frac{1}{x} = \cos\theta - i\sin\theta$

$x - \frac{1}{x} = 2i\sin\theta \quad , x^n - \frac{1}{x^n} = 2i\sin n\theta$

$2i\sin\theta = x - \frac{1}{x}$

$(2i\sin\theta)^5 = (x - \frac{1}{x})^5$

$2^5 i^5 \sin^5\theta = (x - \frac{1}{x})^5$

$2^n i^n \sin^n\theta = (x^n - \frac{1}{x^n}) - nC_1(x^{n-2} - \frac{1}{x^{n-2}}) + nC_2 (x^{n-4} - \frac{1}{x^{n-4}}) - \dots$

$2^5 i^5 \sin^5\theta = (x^5 - \frac{1}{x^5}) - 5C_1(x^3 - \frac{1}{x^3}) + 5C_2(x - \frac{1}{x})$

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$= 2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta)$

$= 2i(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$.

$\sin^5\theta = \frac{2i}{2^5 . i^5}[\sin 5\theta - 5\sin 3\theta + 10\sin\theta]$

$= \frac{1}{2^4 i^4}[\sin 5\theta - 5\sin 3\theta + 10\sin\theta]$

$\sin^5\theta = \frac{1}{2^4}[\sin 5\theta - 5\sin 3\theta + 10\sin\theta]$

3. Expand $\cos^5\theta\sin^3\theta$ in a series of sines of multiples of θ.

Soln:

Let $x = \cos\theta + i\sin\theta$

$\frac{1}{x} = \cos\theta - i\sin\theta$

$x + \frac{1}{x} = 2\cos\theta$; $x^n + \frac{1}{x^n} = 2\cos n\theta$

$x - \frac{1}{x} = 2i\sin\theta$; $x^n - \frac{1}{x^n} = 2i\sin n\theta$

consider,

$2\cos\theta = x + \frac{1}{x}$ $\qquad$ $2i\sin\theta = x - \frac{1}{x}$

$(2\cos\theta)^5 = (x+\frac{1}{x})^5$ $\qquad$ $(2i\sin\theta)^3 = (x-\frac{1}{x})^3$

$2^5\cos^5\theta = (x+\frac{1}{x})^5 \to ①$ $\qquad$ $2^3 i^3\sin^3\theta = (x-\frac{1}{x})^3 \to ②$

$2^5\cos^5\theta\, 2^3 i^3\sin^3\theta = (x+\frac{1}{x})^5(x-\frac{1}{x})^3$

$2^5\cos^5\theta\, i^3\sin^3\theta = (x+\frac{1}{x})^2(x+\frac{1}{x})^3(x-\frac{1}{x})^3$

$-2^5 i\cos^5\theta\sin^3\theta = (x+\frac{1}{x})^2[(x+\frac{1}{x})(x-\frac{1}{3})]^3$

$= (x+\frac{1}{x})^2[x^2 - \frac{1}{x^2}]^3$

$= (x^2+\frac{1}{x^2}+2)(x^6 - \frac{1}{x^6} - 3x^2 + 3)$

$= x^8 - \frac{1}{x^4} - 3x^4 + 3 + x^4 - \frac{1}{x^8} - 3$

$+ \frac{3}{x^4} + 2x^6 - \frac{2}{x^6} - 6x^2 + \frac{6}{x^2}$

$-2^6 i \cos^5\theta \sin^3\theta = (x^8 - \frac{1}{x^8}) + 2(x^6 - \frac{1}{x^6}) - 2(x^4 - \frac{1}{x^4})$
$- 6(x^2 - \frac{1}{x^2})$.

$= 2i \sin 8\theta + 2(2i \sin 6\theta) - 2$
$(2i \sin 4\theta) - 6(2i \sin 2\theta)$.

$= 2i(\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta$
$- 6\sin 2\theta)$.

$\cos^5\theta \sin^3\theta = \frac{-1}{2^7}[\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta$
$- 6\sin 2\theta)$.

Expansion of $\sin\theta$, $\cos\theta$ and $\tan\theta$ in powers of θ.

Theorem:

When θ is expressed in radians.

i) $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$

ii) $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$

iii) $\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \cdots$

Proof:

The Taylor's series expansion of f about the origin is given by $f(\theta) = f(0) + f'(0)\frac{\theta}{1!} + f''(0)\frac{\theta^2}{2!}$
$+ f'''(0)\frac{\theta^3}{3!} + \cdots$

i) Let $f(\theta) = \sin\theta \Rightarrow f(0) = \sin 0 = 0$

$f'(\theta) = \cos\theta \Rightarrow f'(0) = \cos 0 = 1$

$f''(\theta) = -\sin\theta \Rightarrow f''(0) = -\sin 0 = 0$

$f'''(\theta) = -\cos\theta \Rightarrow f'''(0) = -\cos 0 = -1$

$f^{iv}(\theta) = +\sin\theta \Rightarrow f^{iv}(0) = \sin 0 = 0$

$f(\theta) = f(0) + f'(0)\frac{\theta}{1!} + f''(0)\frac{\theta^2}{2!} + f'''(0)\frac{\theta^3}{3!} + \dots$

$\sin\theta = 0 + \frac{\theta}{1!} + 0 - \frac{\theta^3}{3!} + 0 + \frac{\theta^5}{5!} - \dots$

$\sin\theta = \frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$

ii) Let $f(\theta) = \cos\theta = f(0) = \cos 0 = 1$

$f'(\theta) = -\sin\theta = f'(0) = -\sin 0 = 0$

$f''(\theta) = -\cos\theta = f''(0) = -\cos 0 = -1$

$f'''(\theta) = \sin\theta = f'''(0) = \sin 0 = 0$

$f^{iv}(\theta) = \cos\theta = f^{iv}(0) = \cos 0 = 1$

$\vdots$

$f(\theta) = f(0) + f'(0)\frac{\theta}{1!} + f''(0)\frac{\theta^2}{2!} + f'''(0)\frac{\theta^3}{3!} + \dots$

$\cos\theta = 1 + 0 - \frac{\theta^2}{2!} + 0 + \frac{\theta^4}{4!} - \dots$

$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots$

iii) Let $f(\theta) = \tan\theta \Rightarrow f(0) = \tan 0 = 0$

$f'(\theta) = \sec^2\theta \Rightarrow f'(0) = \sec^2 0 = 1$

$f''(\theta) = 2\sec\theta \cdot \sec\theta\tan\theta = 2\sec^2\theta\tan\theta$

$f''(0) = 2\sec^2 0 \tan 0 = 0$

$f'''(\theta) = 2[\sec^2\theta \cdot \sec^2\theta + \tan\theta \cdot 2\sec\theta\sec\theta\tan\theta] \Rightarrow f'''(0) = 2$

$= 2\sec^4\theta + 4\sec^2\theta\tan^2\theta$

$f^{iv}(\theta) = 8\sec^3\theta \sec\theta\tan\theta + 4[\sec^2\theta \cdot 2\tan\theta\sec^2\theta + \tan^2\theta\, 2\sec\theta \cdot \sec\theta\tan\theta]$

$= 8\sec^3\theta\tan\theta + 4[2\sec^4\theta\tan\theta + 2\sec^2\theta\tan^3\theta]$

$= 8\sec^3\theta\tan\theta + 8\sec^4\theta\tan\theta + 8\sec^2\theta\tan^3\theta$

$= 8\sec^3\theta\tan\theta + 8\{\sec^4\theta\tan\theta + 8\sec^2\theta\tan^3\theta$

$f^{iv}(0) = 0$

$f^{v}(\theta) = 8[\sec^4\theta\sec^2\theta + \tan\theta\, 4\sec^3\theta\sec\theta\tan\theta + \tan\theta\, 4\sec^3\theta\sec\theta\tan\theta +$
$\sec^4\theta\sec^2\theta + \sec^2\theta\, 3\tan^2\theta\sec^2\theta + \tan^3\theta\, 2\sec\theta\sec\theta\tan\theta]$

$= 8(\sec^6\theta + 4\sec^4\theta\tan^2\theta + 4\sec^4\theta\tan^2\theta + \sec^6\theta + 3\sec^4\theta\tan^2\theta + 2\tan^4\theta\sec^2\theta]$

$f^{v}(0) = 8(2\sec^6\theta + 8\sec^4\theta\tan^2\theta + 3\tan^2\theta\sec^4\theta + 2\sec^2\theta\tan^4\theta = 16$

$\tan(\theta) = 0 + \frac{\theta}{1!} + 0 + 2 \times \frac{\theta^3}{1\times2\times3} + 0 + 16 \cdot \frac{\theta^5}{1\times2\times3\times4\times5} + \dots$

$\tan(\theta) = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \dots$

1. P.T: $2^4 \cos^5\varphi = \cos 5\varphi + 5\cos 3\varphi + 10\cos\varphi$.

Soln:

Let $x = \cos\varphi + i\sin\varphi$

$\frac{1}{x} = \cos\varphi - i\sin\varphi$

$x + \frac{1}{x} = 2\cos Q \; ; \; x^n + \frac{1}{x^n} = 2\cos nQ$

$(2\cos Q)^5 = \left(x+\frac{1}{x}\right)^5$

$2^5\cos^5\varphi = \left(x^5+\frac{1}{x^5}\right) + 5C_1\left(x^3+\frac{1}{x^3}\right) + 5C_2\left(x+\frac{1}{x}\right)$

$= 2\cos 5\varphi + 5(2\cos 3\varphi) + 10(2\cos\varphi)$

$2^5\cos^5\varphi = 2(\cos 5\varphi + 5\cos 3\varphi + 10\cos\varphi)$

$2^4\cos^5\varphi = \cos 5\varphi + 5\cos 3\varphi + 10\cos\varphi$.

2. P.T: $2^6\cos^7\varphi = \cos 7\varphi + 7\cos 5\varphi + 21\cos 3\varphi + 35\cos\varphi$

Soln:

Let $x = \cos\varphi + i\sin\varphi$

$\frac{1}{x} = \cos\varphi - i\sin\varphi$

$x + \frac{1}{x} = 2\cos\varphi \; ; \; x^n + \frac{1}{x^n} = 2\cos n\varphi$

$(2\cos\varphi)^7 = \left(x+\frac{1}{x}\right)^7$

$2^7\cos^7\varphi = \left(x^7+\frac{1}{x^7}\right) + 7C_1\left(x^5+\frac{1}{x^5}\right) + 7C_2\left(x^3+\frac{1}{x^3}\right) + 7C_3\left(x+\frac{1}{x}\right)$

$= 2\cos 7\varphi + 7(2\cos 5\varphi) + 21(2\cos 3\varphi) + 35(2\cos\varphi)$

$2^7\cos^7\varphi = 2[\cos 7\varphi + 7\cos 5\varphi + 21\cos 3\varphi + 35\cos\varphi]$

$2^6\cos^7\varphi = \frac{1}{2}[\cos 7\varphi + 7\cos 5\varphi + 21\cos 3\varphi + 35\cos\varphi]$

3. P.T: $\sin^9\varphi = \frac{1}{256}[\sin 9\varphi - 9\sin 7\varphi + 36\sin 5\varphi - 84\sin 3\varphi + 126\sin\varphi]$.

soln:

$$x = \cos\varphi + i\sin\varphi$$

$$\frac{1}{x} = \cos\varphi - i\sin\varphi$$

$$x - \frac{1}{x} = 2i\sin\varphi \ ; \ x^n - \frac{1}{x^n} = 2i\sin n\varphi$$

$$(2i\sin\varphi)^9 = \left(x - \frac{1}{x}\right)^9$$

$$2^9 i^9 \sin^9\varphi = \left(x^9 - \frac{1}{x^9}\right) - 9C_1\left(x^7 - \frac{1}{x^7}\right) + 9C_2\left(x^5 - \frac{1}{x^5}\right) - 9C_3\left(x^3 - \frac{1}{x^3}\right) + 9C_4\left(x + \frac{1}{x}\right).$$

$$= 2i\sin 9\varphi - 9(2i\sin 7\varphi) + 36(2i\sin 5\varphi) - 84(2i\sin 3\varphi) + 126(2i\sin\varphi).$$

$$2^9 i^9 \sin^9\varphi = 2i[\sin 9\varphi - 9\sin 7\varphi + 36\sin 5\varphi - 84\sin 3\varphi + 126\sin\varphi]$$

$$= \frac{1}{2^8 i^8}[\sin 9\varphi - 9\sin 7\varphi + 36\sin 5\varphi - 84\sin 3\varphi + 126\sin\varphi].$$

$$= \frac{1}{256}[\sin 9\varphi - 9\sin 7\varphi + 36\sin 5\varphi - 84\sin 3\varphi + 126\sin\varphi]$$

4. P.T: $\sin^3\theta\cos\theta = -\frac{1}{8}[\sin 4\theta - 2\sin 2\theta]$

soln:

Let $x = \cos\theta + i\sin\theta$

$$\frac{1}{x} = \cos\theta - i\sin\theta$$

$$x + \frac{1}{x} = 2\cos\theta \ ; \ x^n + \frac{1}{x^n} = 2\cos n\theta$$

$$x - \frac{1}{x} = 2i\sin\theta \ ; \ x^n - \frac{1}{x^n} = 2i\cos n\theta.$$

Consider,

$$2\cos\theta = x + \frac{1}{x} \qquad 2i\sin\theta = x - \frac{1}{x}$$

$$2^3 i^3 \sin^3\theta = \left(x - \frac{1}{x}\right)^3$$

$$2\cos\theta\, 2^3 i^3 \sin^3\theta = \left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)^3$$

$$=\left(x+\frac{1}{x}\right)\left(x^3-\frac{1}{x^3}-\frac{3x^2}{x}+\frac{3x}{x^2}\right)$$

$$=\left(x+\frac{1}{x}\right)\left(x^3-\frac{1}{x^3}-3x+\frac{3}{x}\right)$$

$$=\left(x^4-\frac{1}{x^2}-3x^2+3+x^2-\frac{1}{x^4}-3+\frac{3}{x^2}\right)$$

$$=\left(x^4-\frac{1}{x^4}\right)+2\left(x^2-\frac{1}{x^2}\right)$$

$$=2i\sin4\theta+2(2i\sin2\theta)$$

$$=2i(\sin4\theta+2\sin2\theta).$$

$$2^4 i^3\sin^3\theta\cos\theta = 2i(\sin4\theta+2\sin2\theta)$$

$$\sin^3\theta\cos\theta = -\frac{1}{8}(\sin4\theta+2\sin2\theta)$$

5. P.T: $2^6\sin^4\theta\cos^3\theta = \cos7\theta-\cos5\theta-3\cos3\theta+3\cos\theta$

<u>soln</u>:

$$x=\cos\theta+i\sin\theta$$

$$\frac{1}{x}=\cos\theta-i\sin\theta$$

$x+\frac{1}{x}=2\cos\theta$; $x^n+\frac{1}{x^n}=2\cos n\theta$

$x-\frac{1}{x}=2i\sin\theta$; $x^n-\frac{1}{x^n}=2i\sin n\theta$

consider,

$2\cos\theta = x+\frac{1}{x}$ $\qquad$ $2i\sin\theta = x-\frac{1}{x}$

$(2\cos\theta)^3=\left(x+\frac{1}{x}\right)^3$ $\qquad$ $(2i\sin\theta)^4=\left(x-\frac{1}{x}\right)^4$

$2^3\cos^3\theta=\left(x+\frac{1}{x}\right)^3$ $\qquad$ $2^4 i^4\sin^4\theta=\left(x-\frac{1}{x}\right)^4$

$$2^7 i^4\cos^3\theta\sin^4\theta=\left(x+\frac{1}{x}\right)^3\left(x-\frac{1}{x}\right)^4$$

$$=\left(x+\frac{1}{x}\right)\left[\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)\right]^2$$

$$=\left(x+\frac{1}{x}\right)\left[x^2-\frac{1}{x^2}\right]^2$$

$2^7 i^4 \cos^3\theta \sin^4\theta = (x+\frac{1}{x})(x^4 - 2 + \frac{1}{x^4})$

$-\frac{1}{x^2} \quad -3x^2 \quad x^2$

$= x^5 - 2x + \frac{1}{x^3} + x^3 - \frac{2}{x} + \frac{1}{x^5}$

$\frac{2}{x^2} \quad 2x^2$

$= (x^5 + \frac{1}{x^5}) + (x^3 + \frac{1}{x^3}) - (x + \frac{1}{x})$

$= 2\cos 5\theta + 2\cos 3\theta$

$2^7 i^4 \cos^3 \sin^4\theta = (x+\frac{1}{x})^3 (x-\frac{1}{x})^4$

$= [(x+\frac{1}{x})(x-\frac{1}{x})]^3 (x-\frac{1}{x})$

$= [x^2 - \frac{1}{x^2}]^3 (x-\frac{1}{x})$

$= (x^6 - 3x^2 + 3\frac{1}{x^2} - \frac{1}{x^6})(x-\frac{1}{x})$

$= x^7 - 3x^3 + \frac{3}{x} - \frac{1}{x^5} - x^5 + 3x - 3\frac{1}{x^3} + \frac{1}{x^7}$

$= (x^7 + \frac{1}{x^7}) - (x^5 + \frac{1}{x^5}) - 3(x^3 + \frac{1}{x^3}) + 3(x + \frac{1}{x})$

$= 2\cos 7\theta - 2\cos 5\theta - 3\cos 3\theta + 3\cos\theta$

$2^7 i^4 \cos^3\theta \sin^4\theta = 2\cos 7\theta - 2\cos 5\theta - 3\cos 3\theta + 3\cos\theta$

$2^6 \sin^4\theta \cos^3\theta = \cos 7\theta - 2\cos 5\theta - 3\cos 3\theta + 3\cos\theta$

1. Find approximately the value of θ in radians if $\frac{\sin\theta}{\theta} = \frac{863}{864}$

<u>soln:</u>

$$\sin\theta = \frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^4}{4!} - \dots$$

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \dots$$

$$\frac{863}{864} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{4!} - \dots$$

$$1 - \frac{1}{864} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{4!} - \dots$$

$$-\frac{1}{864} = -\frac{\theta^2}{3!} + \frac{\theta^4}{4!} - \dots$$

$$-\frac{1}{864} \approx -\frac{\theta^2}{3!} \text{ (neglecting the higher powers of } \theta)$$

$$\frac{\theta^2}{6} \approx \frac{1}{864}$$

$$\theta^2 \approx \frac{6}{864}$$

$$\theta^2 \approx \frac{1}{144}$$

$$\theta \approx \frac{1}{12} \text{ radians.}$$

2. If $\frac{\tan\theta}{\theta} = \frac{2524}{2523}$. S.T. θ is approximately equal to $1°58'$.

<u>Soln:</u>

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \dots$$

$$\frac{\tan\theta}{\theta} = 1 + \frac{\theta^2}{3} + \frac{2\theta^4}{15} + \dots$$

$$\frac{2524}{2523} = 1 + \frac{\theta^2}{3} + \frac{2\theta^4}{15} + \dots$$

$$1 + \frac{1}{2523} = 1 + \frac{\theta^2}{3} + \frac{2\theta^4}{15} + \dots$$

$$\frac{1}{2523} = \frac{\theta^2}{3} + \frac{2\theta^4}{15} + \dots$$

$$\frac{1}{2523} \approx \frac{\theta^2}{3} \text{ (Neglecting the higher powers of } \theta)$$

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$\theta^2 \approx \frac{3}{2523}$

$\theta^2 \approx \frac{1}{841}$

$\theta \approx \frac{1}{29}$ radians

$\approx \frac{1}{29} \times \frac{180^\circ}{\pi}$

$\approx \frac{1}{29} \times \frac{180^\circ}{22} \times 7$

$\approx \frac{1}{29} \times 57.29$ degrees approximately.

$\theta \approx 1^\circ 58'$

(ii) $\theta = 1^\circ 58'$ (approximately).

1. (ii) Find the S.T. if x is small $\cos(\alpha+x) = \cos\alpha - x\sin\alpha + \frac{x^2}{2}\cos\alpha + \frac{x^3}{6}\sin\alpha$

Soln:

$\cos(\alpha+x) = \cos\alpha\cos x - \sin\alpha\sin x$

$= \cos\alpha\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) - \sin\alpha\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$

$= \cos\alpha - \frac{x^2}{2!}\cos\alpha + \frac{x^4}{4!}\cos\alpha - \cdots - x\sin\alpha + \frac{x^3}{3!}\sin\alpha - \frac{x^5}{5!}\sin\alpha + \cdots$

$\cos(\alpha+x) = \cos\alpha - x\sin\alpha - \frac{x^2}{2}\cos\alpha + \frac{x^3}{6}\sin\alpha$

$\because$ since x is small & neglecting the higher powers of x

2. Solve approximately $\cos\left(\frac{\pi}{3}+\theta\right) = 0.49$

Soln:

$\cos\left(\frac{\pi}{3}+\theta\right) = 0.49$

$\cos\frac{\pi}{3}\cos\theta - \sin\frac{\pi}{3}\sin\theta = 0.49$

$\frac{1}{2}\left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right] - \frac{\sqrt{3}}{2}\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right] = 0.49$

$\frac{1}{2} - \frac{\sqrt{3}}{2}\theta = 0.49$

$-\frac{\sqrt{3}}{2}\theta = 0.49 - \frac{1}{2}$

$$\frac{\sqrt{3}}{2}\theta = \frac{1}{2} - 0.49$$

$$= \frac{1-0.98}{2} = \frac{0.02}{2}$$

$$= 0.01$$

$$= \frac{1}{100}$$

$$\theta = \frac{1}{100} \times \frac{2}{\sqrt{3}}$$

$$= \frac{1}{50\sqrt{3}}$$

$$= \frac{\sqrt{3}}{150}$$

$$= \frac{1.732}{150}$$

$\theta = 0.0115$ radians

$\theta = 40$ minutes (approximately)

3. Evaluate $\sin 3^\circ$ correct to 3 places of decimals.

Soln:

$180^\circ = \pi$ radian

$1^\circ = \frac{\pi}{180}$ radian

$$3^\circ = \frac{3\pi}{180}$$

$$3^\circ = \frac{\pi}{60}$$

$$\sin 3^\circ = \sin \pi/60$$

$$= \frac{\pi}{60} - \frac{(\pi/60)^3}{3!} + \frac{(\pi/60)^5}{5!} - \cdots$$

$= \frac{\pi}{60}$ (Neglecting the higher powers of $\left(\frac{\pi}{60}\right)$)

$$= \frac{22}{7\times 60}$$

$$\sin 3^\circ = 0.052$$

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* 4. If $x = \frac{2}{1!} - \frac{4}{3!} + \frac{6}{5!} - \frac{8}{7!} + \cdots$ and
VQ $y = 1 + \frac{2}{1!} - \frac{2^3}{3!} + \frac{2^5}{5!} - \cdots$ P.T. $x^2 = y$

Soln:

$$x = \frac{2}{1!} - \frac{4}{3!} + \frac{6}{5!} - \frac{8}{7!} + \cdots$$

$$= \frac{(1+1)}{1!} - \frac{(3+1)}{3!} + \frac{(5+1)}{5!} - \frac{(7+1)}{7!} + \cdots$$

$3! = 1\times2\times3$

$5! = 1\times2\times3\times4\times5$

$$= \frac{1}{1!} + \frac{1}{1!} - \frac{3}{3!} - \frac{1}{3!} + \frac{5}{5!} + \frac{1}{5!} - \frac{7}{7!} - \frac{1}{7!} + \cdots$$

$$= \left(\frac{1}{1!} - \frac{3}{3!} + \frac{5}{5!} - \frac{7}{7!}\right) + \left(\frac{1}{1!} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots\right)$$

$$= \left(\frac{1}{1!} - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!}\right) + \left(\frac{1}{1!} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots\right)$$

$$x = \cos(1) + \sin 1$$

$$x^2 = (\cos 1 + \sin 1)^2$$

$$= \cos^2 1 + \sin^2 1 + 2\cos 1 \sin 1$$

$$= 1 + \sin 2$$

$$= 1 + \left(2 - \frac{2^3}{3!} + \frac{2^5}{5!} - \cdots\right)$$

$$= 1 + \frac{2}{1!} - \frac{2^3}{3!} + \frac{2^5}{5!} - \cdots$$

$$x^2 = y$$

5. If θ is small. P.T. $\theta \cot\theta = 1 - \frac{\theta^2}{3} - \frac{\theta^4}{45}$ approximately

Soln:

$$\theta \cot\theta = \frac{\theta}{\tan\theta}$$

$$= \frac{\theta}{\theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \cdots}$$

$$= \frac{\theta}{\theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15}}$$ (Neglecting higher powers of θ)

$$= \frac{\theta}{\theta\left(1 + \frac{\theta^2}{3} + \frac{2\theta^4}{15}\right)}$$

$$\theta \cot\theta = \frac{1}{\left(1+\frac{\theta^2}{3}+\frac{2\theta^4}{15}\right)} \qquad (1+x)^{-1} = 1-x+x^2-x^3+\ldots$$

$$= \left(1+\frac{\theta^2}{3}+\frac{2\theta^4}{15}\right)^{-1}$$

$$= 1-\left(\frac{\theta^2}{3}+\frac{2\theta^4}{15}\right)+\left(\frac{\theta^2}{3}+\frac{2\theta^4}{15}\right)^2 - \ldots$$

$$= 1-\frac{\theta^2}{3}-\frac{2\theta^4}{15}+\frac{\theta^4}{9}+\frac{4\theta^8}{225}+\frac{2\theta^2}{3}\frac{2\theta^4}{15}\ldots$$

$$= 1-\frac{\theta^2}{3}-\frac{2\theta^4}{15}+\frac{\theta^4}{9}+\frac{4\theta^8}{225}+\frac{4\theta^6}{45}\ldots$$

$$= 1-\frac{\theta^2}{3}-\frac{2\theta^4}{15}+\frac{\theta^4}{9} \quad \text{(N. h.P of } \theta)$$

$$= 1-\frac{\theta^2}{3}-\left(\frac{2}{15}-\frac{1}{9}\right)\theta^4$$

$$= 1-\frac{\theta^2}{3}-\left(\frac{6-5}{45}\right)\theta^4$$

$$\theta\cot\theta = 1-\frac{\theta^2}{3}-\frac{\theta^4}{45}$$

6. P.T. When θ is small,

$$\frac{1}{6}\sin^3\theta = \frac{\theta^3}{3!}-(1+3^2)\frac{\theta^5}{5!}+(1+3^2+3^4)\frac{\theta^7}{7!}-\ldots$$

Soln:

$$\sin^3\theta = \frac{3\sin\theta-\sin 3\theta}{4}$$

$$4\sin^3\theta = 3\sin\theta-\sin 3\theta$$

$$= 3\left[\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}+\ldots\right]-\left[3\theta-\frac{(3\theta)^3}{3!}+\frac{(3\theta)^5}{5!}-\frac{(3\theta)^7}{7!}+\ldots\right]$$

$$= 3\theta-\frac{3\theta^3}{3!}+\frac{3\theta^5}{5!}-\frac{3\theta^7}{7!}+\ldots-3\theta+\frac{(3\theta)^3}{3!}-\frac{(3\theta)^5}{5!}+\frac{(3\theta)^7}{7!}\ldots$$

$$= -\frac{\theta^3}{3!}(3-3^3)+\frac{\theta^5}{5!}(3-3^5)-\frac{\theta^7}{7!}(3-3^7)\ldots$$

$$= 3\left[-\frac{\theta^3}{3!}(1-3^2)+\frac{\theta^5}{5!}(1-3^4)-\frac{\theta^7}{7!}(1-3^6)\ldots\right]$$

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