28.09.18
Friday

UNIT-II

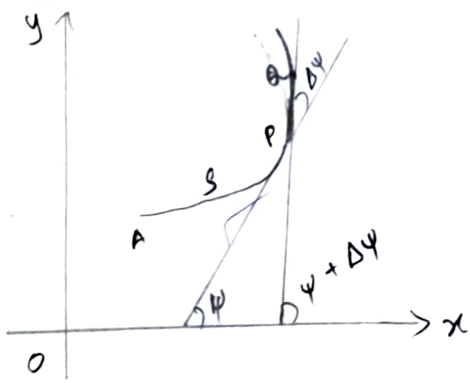
CURVATURE AND RADIUS OF CURVATURE

Curvature :-

* The curve at a given point on the curve determined by the tangent at that point

* The tangent line rotates as the point moves along the curve.

* The curvature of the curve at a given point is defined as the rate of change of bending of the curve at the point.



* Let P be a point on the curve $y = f(x)$

* Q be a neighbouring point on the curve.

* Let the tangent to the curve at P and Q makes angle ψ & $\psi + \Delta\psi$ with

positive direction of x-axis respectively.

* The angle $\Delta\varphi$ between the two tangents depends on the extent of bending of the arc PQ.

* The curvature of the curve at P is defined by the rate of change of φ with respect to s.

* W.K.T the rate of change is measured by $d\varphi/ds$

Radius of Curvature :-

* The reciprocal of the curvature is called the radius of curvature and it is denoted by ρ

$\therefore \rho = ds/d\varphi$ and the curvature at P is

$$1/\rho = \frac{d\varphi}{ds}$$

Cartesian formulae for Radius of curvature :-

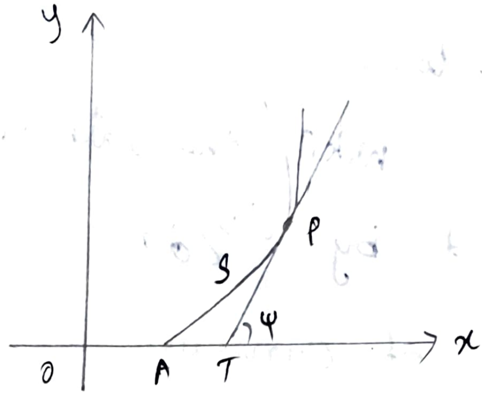
Let P be any point on the curve $y = f(x)$ and PT be the tangent making angle φ with x-axis. Then

W.K.T. $dy/dx = \tan\varphi$

diff. w.r.t. 'x'

$$d^2y/dx^2 = \sec^2\varphi \cdot d\varphi/dx$$

$$= \sec^2\varphi \cdot \frac{d\varphi}{ds} \cdot \frac{ds}{dx}$$



$$\therefore ds/d\varphi = \frac{\sec^2\varphi}{d^2y/dx^2} \cdot ds/dx$$

$$= \frac{(1+\tan^2\varphi)\sec\varphi}{d^2y/dx^2} \quad \left[\because \frac{dx}{ds} = \cos\varphi\right]$$

$$= \frac{(1+\tan^2\varphi)(\sqrt{1+\tan^2\varphi})}{d^2y/dx^2}$$

$$\rho = \frac{(1+\tan^2\varphi)(1+\tan^2\varphi)^{1/2}}{d^2y/dx^2}$$

$$\rho = \frac{(1+\tan^2\varphi)^{3/2}}{d^2y/dx^2}$$

$$\rho = \frac{\left(1+\left(dy/dx\right)^2\right)^{3/2}}{d^2y/dx^2} \quad (\text{or}) \quad \rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

where, $y_1 = dy/dx$

$y_2 = d^2y/dx^2$

1. Find the radius of curvature at the point $(a/4, a/4)$ to the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$.

Soln:

$$\rho = \frac{\left(1+\left(dy/dx\right)^2\right)^{3/2}}{d^2y/dx^2}$$

Given eqn of the curve.

$$\sqrt{x}+\sqrt{y}=\sqrt{a} \rightarrow ①$$

$$x^{1/2}+y^{1/2}=a^{1/2}$$

Diff. w.r.t 'x'

$$\tfrac{1}{2}x^{1/2-1}+\tfrac{1}{2}y^{1/2-1}\cdot dy/dx=0$$

$$\tfrac{1}{2}x^{-1/2}+\tfrac{1}{2}y^{-1/2}\cdot dy/dx=0$$

$$\tfrac{1}{2\sqrt{x}}+\tfrac{1}{2\sqrt{y}}\,dy/dx=0$$

$$\tfrac{1}{2\sqrt{y}}\cdot dy/dx=-\tfrac{1}{2\sqrt{x}}$$

$$dy/dx=-\frac{2\sqrt{y}}{2\sqrt{x}}$$

$$dy/dx=-\sqrt{y}/\sqrt{x} \rightarrow ②$$

$$dy/dx\,(a/4, a/4)=\frac{-\sqrt{a/4}}{\sqrt{a/4}}$$

$$\boxed{dy/dx=-1}$$

$$d(u/v)=\frac{vu'-uv'}{v^2}$$

② ⇒ diff. w.r.t 'x'

$$\frac{d^2y}{dx^2} = -\left[\frac{\sqrt{x}\left(\frac{1}{2} y^{-1/2} \frac{dy}{dx}\right) - \sqrt{y}\ \frac{1}{2} x^{-1/2}(1)}{x}\right]$$

$$= -\left[\frac{\frac{\sqrt{x}}{2\sqrt{y}} \cdot \frac{dy}{dx} - \frac{\sqrt{y}}{2\sqrt{x}}}{x}\right]$$

$$= -\left[\frac{\frac{-\sqrt{x}}{2\sqrt{y}} - \frac{\sqrt{y}}{2\sqrt{x}}}{x}\right]$$

$$\frac{d^2y}{dx^2}\left(\frac{a}{4}, \frac{a}{4}\right) = \frac{\frac{\sqrt{a/4}}{2\sqrt{a/4}} + \frac{\sqrt{a/4}}{2\sqrt{a/4}}}{a/4}$$

$$= \frac{\frac{1}{2} + \frac{1}{2}}{a/4}$$

$$\frac{d^2y}{dx^2} = \frac{4}{a}$$

$$\rho = \frac{\left[1 + (-1)^2\right]^{3/2} a}{4}$$

$$= \frac{(1+1)^{3/2} a}{4}$$

$$= \frac{2^{3/2} a}{4}$$

$$= \frac{2\sqrt{2}\, a}{4}$$

$$\boxed{\rho = \frac{a\sqrt{2}}{2}}$$

2. Find the radius of curvature at $x = y = 3a/2$ to the curve $x^3 + y^3 = 3axy$

Soln:
$$\rho = \frac{\left(1 + (dy/dx)^2\right)^{3/2}}{d^2y/dx^2}$$

Given eqn of the curve,

$$x^3 + y^3 = 3axy \rightarrow (1)$$

$$3x^2 + 3y^2 \, dy/dx = 3a\left[x \, dy/dx + y\right]$$

$$3\left[x^2 + y^2 \cdot dy/dx\right] = 3a\left[x \cdot dy/dx + y\right]$$

$$x^2 + y^2 \, dy/dx = ax \, dy/dx + ay$$

$$y^2 \, dy/dx - ax \, dy/dx = ay - x^2$$

$$dy/dx\left[y^2 - ax\right] = ay - x^2$$

$$dy/dx = \frac{ay - x^2}{y^2 - ax}$$

$$dy/dx\left(3a/2, 3a/2\right) = \frac{\frac{3a^2}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - \frac{3a^2}{2}}$$

$$= \frac{\frac{3a^2}{2}\left(1 - 3/2\right)}{\frac{3a^2}{2}\left(3/2 - 1\right)}$$

$$= \frac{-1/2}{1/2}$$

$$\boxed{dy/dx = -1}$$

diff. w.r.t. 'x'

$$\frac{d^2y}{dx^2} = \frac{(y^2-ax)\left(a\,dy/dx - 2x\right) - (ay-x^2)\left(2y \cdot dy/dx - a\right)}{(y^2-ax)^2}$$

$$= \frac{\left[\frac{9a^2}{4} - \frac{3a^2}{2}\right]\left[-a-3a\right] - \left[\frac{3a^2}{2} - \frac{9a^2}{4}\right]\left[-3a-a\right]}{\left[\frac{9a^2}{4} - \frac{3a^2}{2}\right]^2}$$

$$= \frac{\cancel{\left[\frac{9a^2}{4} - \frac{3a^2}{2}\right]}\left[-a-3a-3a-a\right]}{\left[\frac{9a^2}{4} - \frac{3a^2}{2}\right]^{\cancel{2}}}$$

$$= \frac{-8a}{\frac{9a^2}{4} - 3a^2/2}$$

$$= \frac{-8a}{\frac{9a^2-6a^2}{4}}$$

$$= \frac{-8\cancel{a}}{3a^{\cancel{2}}} \times 4$$

$$\frac{d^2y}{dx^2} = -\frac{32}{3a}$$

$$\rho = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}{d^2y/dx^2}$$

$$\rho = \frac{\left(1+(-1)^2\right)^{3/2}}{-32/3a}$$

$$\rho = \frac{(1+1)^{3/2}}{-32/3a}$$

$$\rho = \frac{(2)^{3/2}}{-32/3a}$$

$$\rho = \frac{\cancel{2}\sqrt{2}\cdot 3a}{-\cancel{32}\ 16}$$

$$\rho = \frac{-\sqrt{2}\cdot 3a}{16}$$

$$\boxed{\rho = \frac{+3\sqrt{2}\cdot a}{16}}$$

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Monday.

3. S.T. the radius of curvature at $(a,0)$ on the curve $y^2 = \frac{a^2(a-x)}{x}$ is $a/2$

Soln :-

Given curve,

$$y^2 = \frac{a^2(a-x)}{x}$$

$$2y\,{dy}/{dx} = \frac{x(a^2(-1)) - a^2(a-x)(1)}{x^2}$$

$$= \frac{-a^2x - a^3 + a^2x}{x^2}$$

$$2y \cdot {dy}/{dx} = -a^3/x^2$$

$$dy/dx = -\frac{a^3}{2yx^2}$$

$$dy/dx(a,0) = \frac{-a^3}{2(0)a^2}$$

$$\boxed{dy/dx = \infty} \quad \infty = 1/0$$

$$dx/dy = 0$$

diff. w.r.t. 'x'

$$d^2y/dx^2 = \frac{2yx^2(0) - a^3(2\,dy/dx \cdot 2x)}{4x^4y^2}$$

consider,

$$dx/dy = -\frac{2x^2y}{a^3}$$

$$\frac{d^2x}{dy^2} = -2/a^3\left[x^2(1) + y\,2x\,\frac{dx}{dy}\right]$$

$$= -\frac{2}{a^3}\left[x^2 + 2xy\,\frac{dx}{dy}\right]$$

$$\frac{d^2x}{dy^2}(a,0) = -\frac{2}{a^3}\left[a^2 + 0\right]$$

$$= -\frac{2}{a^3} \times a^2$$

$$\frac{d^2y}{dx^2} = -\frac{2}{a}$$

$$\rho = \frac{\left(1 + \left(\frac{dx}{dy}\right)^2\right)^{3/2}}{d^2x/dy^2}$$

$$= \frac{(1+0)^{3/2}}{-2/a}$$

$$= \frac{1}{-2/a}$$

$$\rho = -a/2$$

$$\boxed{\rho = a/2}$$

03.10.18
Wednesday

4. Find the radius of curvature at the point 'θ'. on the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$.

Soln:-

Given eqn, $x = a(\cos\theta + \theta\sin\theta)$

diff with respect to 'θ'

$dx/d\theta = a(-\sin\theta + \theta\cos\theta + \sin\theta)$

$dx/d\theta = a\theta\cos\theta$

$dx = a\theta\cos\theta\, d\theta \rightarrow (1)$

diff w.r.t. 'θ'

$dy/d\theta = a(\cos\theta - [(\theta)(-\sin\theta) + \cos\theta]$

$= a[\cos\theta + \theta\sin\theta - \cos\theta]$

$dy/d\theta = a\theta\sin\theta$

$dy = a\theta\sin\theta\, d\theta \rightarrow (2)$

$(2)/(1) \Rightarrow dy/dx = \dfrac{a\theta\sin\theta}{a\theta\cos\theta}$

$\boxed{dy/dx = \tan\theta}$

Diff. w.r.t. 'x'

$d^2y/dx^2 = \sec^2\theta \cdot d\theta/dx$

$= \sec^2\theta \cdot 1/a\theta\cos\theta$

$\Rightarrow \sec^3\theta \cdot \dfrac{1}{a\theta}$

$$\frac{d^2y}{dx^2} = \frac{1}{a\theta}\left(\sec^3\theta\right)$$

$$\rho = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}{\frac{d^2y}{dx^2}}$$

$\because 1+\tan^2\theta = \sec^2\theta$

$$\rho = \frac{\left(1+(\tan\theta)^2\right)^{3/2}}{\sec^3\theta}\, a.\theta$$

$$= \frac{(\sec^2\theta)^{3/2}}{\sec^3\theta}\, a.\theta$$

$$= \frac{\sec^3\theta}{\sec^3\theta}\, a.\theta$$

$$\boxed{\rho = a.\theta}$$

5. P.T. the radius of curvature at the point $(a\cos^3\theta, a\sin^3\theta)$ on the curve $x^{2/3}+y^{2/3}=a^{2/3}$ is $3a\sin\theta\cos\theta$.

Soln:-

Given eqn

$$x^{2/3}+y^{2/3}=a^{2/3}$$

$$x = a\cos^3\theta$$

$$\cancel{\frac{dx}{d\theta} = a^3(-\sin\theta)}$$

$$\cancel{\frac{dx}{d\theta} = -\sin\theta\, a^3}$$

$dx/d\theta = a\,3\cos^2\theta \cdot (-\sin\theta)$

$dx/d\theta = -a\,3\cos^2\theta \cdot \sin\theta,$

$dx = -3a\cos^2\theta\,\sin\theta \cdot d\theta \rightarrow ①$

$y = a\sin^3\theta$

$dy/d\theta = a\,3\sin^2\theta\,(\cos\theta)$

$dy/d\theta = 3a\sin^2\theta\cos\theta$

$dy = 3a\sin^2\theta\cos\theta\,d\theta \rightarrow ②$

$②/① \Rightarrow dy/dx = \dfrac{\cancel{3a}\sin^{\cancel{2}}\theta\cancel{\cos\theta}\cancel{d\theta}}{-\cancel{3a}\cos^{\cancel{2}}\theta\sin\cancel{\theta}\,\cancel{d\theta}}$

$dy/dx = -\dfrac{\sin\theta}{\cos\theta}$

$dy/dx = -\tan\theta$

diff. w.r.t. x

$d^2y/dx^2 = -\sec^2\theta \cdot \dfrac{1}{-a\,3\cos^2\theta\sin\theta}$

$\sec^4\theta\,\dfrac{1}{a\,3\sin\theta}$

$$\frac{d^2y}{dx^2} = \frac{1}{3a}\left[\frac{\sec^4\theta}{\sin\theta}\right)$$

$$\rho = \frac{\left(1+\left(dy/dx\right)^2\right)^{3/2}}{d^2y/dx^2}$$

$$\rho = \frac{\left(1+(-\tan\theta)^2\right)^{3/2}}{\sec^4\theta\left(\frac{1}{3a\sin\theta}\right)}$$

$$= \frac{\left(1+\tan^2\theta\right)^{3/2}}{\frac{1}{3a}\frac{\sec^4\theta}{\sin\theta}}$$

$$= \frac{\left(\sec^2\theta\right)^{3/2}}{\frac{1}{3a}\frac{\sec^4\theta}{\sin\theta}}$$

$$= \frac{\sec^3\theta}{\frac{1}{3a}\frac{\sec^4\theta}{\sin\theta}}$$

$$= \frac{1}{\sec\theta} \times \sin\theta\, 3a.$$

$$= 3a \times \frac{1}{\sec\theta} \times \sin\theta$$

$$\rho = 3a\cos\theta\sin\theta$$

6. Find the radius of curvature at the point θ on the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$

Soln:-

$$x = a(\theta + \sin\theta)$$

$$dx/d\theta = a(1 + \cos\theta) \rightarrow (1)$$

$$dx = \cancel{adacos}\ a(1+\cos\theta)\, d\theta$$

$$y = a(1 - \cos\theta)$$

$$dy/d\theta = a(\sin\theta) \rightarrow (2) \quad 4a \frac{\cos\theta}{2}$$

$$\div \frac{(2)}{(1)} \Rightarrow dy/dx = \frac{a \sin\theta}{a(1+\cos\theta)}$$

$$\boxed{dy/dx = \frac{\sin\theta}{1+\cos\theta}}$$

$1 + \cos\theta = 2\cos^2 \theta/2$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\sin\theta = 2 \sin\theta/2 \cos\theta/2$

$$dy/dx = \frac{2 \sin\theta/2\ \cancel{\cos\theta/2}}{2\cos^{\cancel{2}}\theta/2}$$

$$dy/dx = \frac{\cancel{2}\sin\theta/2}{\cos\theta/2}$$

$$dy/dx = \tan \theta/2$$

diff. w.r.t. 'x'

$$d^2y/dx^2 = \sec^2 \theta/2 \,(1/2)\, d\theta/dx$$

$$d^2y/dx^2 = 1/2 \sec^2 \theta/2 \frac{1}{a(1+\cos\theta)}$$

$$d^2y/dx^2 = 1/2 \sec^2 \theta/2 \frac{1}{a\cdot(2\cos^2\theta/2)}$$

$$d^2y/dx^2 = \frac{1}{4a} \sec^4 \theta/2$$

$$\rho = \frac{\left(1 + (\tan \theta/2)^2\right)^{3/2}}{1/4a \sec^4 \theta/2}$$

$$\rho = \frac{(1 + \tan^2\theta/2)^{3/2}}{\sec^4 \theta/2} 4a.$$

$$\rho = \frac{(\sec^2\theta/2)^{3/2}}{\sec^4\theta/2} 4a$$

$$\rho = \frac{\sec^3 \theta/2}{\sec^4 \theta/2} 4a$$

$= \frac{1}{\sec \theta/2} \cdot 4a$

$$\boxed{\rho = \cos \theta/2 \cdot 4a}$$

7. S.T. the radius of curvature at 't' on the curve is $x = 6t^2 - 3t^4$, $y = 8t^3$ is $6(1+t^2)^2$.

Soln:-

$x = 6t^2 - 3t^4$

$y = 8t^3$

$dx/dt = 12t - 12t^3$

$dx/dt = 12t(1-t^2)$

$dx = 12t(1-t^2)\,dt \rightarrow (1)$

$dy/dt = 24t^2$

$dy = 24t^2\,dt \rightarrow (2)$

$$dy/dx = \frac{24t^2\,dt}{12t(1-t^2)\,dt}$$

$$dy/dx = \frac{2t}{(1-t^2)} \rightarrow (3)$$

$$d^2y/dx^2 = \frac{(1-t^2)(2)\frac{dt}{dx} - (+2t)(-2t)\frac{dt}{dx}}{(1-t^2)^2}$$

$$d^2y/dx^2 = \frac{(2-2t^2+4t^2)\,dt/dx}{(1-t^2)^2}$$

$$d^2y/dx^2 = \frac{(2+2t^2)}{(1-t^2)^2\, 12t\,(1-t^2)}$$

$$= \frac{\cancel{2}^{1}(1+t^2)}{6\,\cancel{12}\,t\,(1-t^2)^3} = \frac{(1+t^2)}{6t(1-t^2)^3}$$

$$d^2y/dx^2 = \frac{(1+t^2)}{6t(1-t^2)^3}$$

$$\rho = \frac{\left(1+(dy/dx)^2\right)^{3/2}}{d^2y/dx^2}$$

$$= \frac{\left(1+\left(\frac{2t}{(1-t^2)}\right)^2\right)^{3/2}}{\frac{1+t^2}{6t(1-t^2)^3}}$$

=

8. S.T. the radius of curvature at the point (x, y) on the curve $y = c\cosh(x/c)$ is y^2/c.

Soln:

$$dy/dx = -c \sinh(x/c) \, 1/c$$

$$dy/dx = -\sinh x/c$$

$$d^2y/dx^2 = -\cosh(x/c) \cdot 1/c$$

$$= -1/c \cosh(x/c)$$

$$\rho = \frac{[1 + \sinh^2(x/c)]^{3/2}}{-1/c \cosh(x/c)}$$

$$\rho = \frac{c\,[\cosh^2(x/c)]^{3/2}}{-\cosh(x/c)}$$

$$\rho = \frac{c\,(\cosh^3 x/c)}{-\cosh(x/c)}$$

$$\rho = c \cosh^2(x/c) \rightarrow ①$$

$$y/c = \cosh(x/c)$$

$$y^2/c^2 = \cosh^2(x/c) \rightarrow ②$$

① in ②

$$① \Rightarrow \rho = c \cdot (y^2/c^2)$$

$$\rho = y^2/c$$

9. The tangent at two points P & Q on the curve $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ are at right angles. S.T. if ρ_1, ρ_2 be the radii of curvature at these points then $\rho_1^2 + \rho_2^2 = 16a^2$

soln:-

$x = a(\theta - \sin\theta)$ $\quad y = a(1-\cos\theta)$

$dx/d\theta = a(1-\cos\theta)$ $\quad dy/d\theta = a\sin\theta$

$$dy/dx = \frac{a\sin\theta}{a(1-\cos\theta)}$$

$$= \frac{\sin\theta}{1-\cos\theta}$$

$$= \frac{2\sin(\theta/2)\cdot\cos(\theta/2)}{2\sin^2(\theta/2)}$$

$$= \frac{\cos(\theta/2)}{\sin\theta/2}$$

$$dy/dx = \cot\theta/2 \rightarrow (1)$$

$$d^2y/dx^2 = -\csc^2(\theta/2)(1/2)\,d\theta/dx$$

$$d^2y/dx^2 = -1/2\,\csc^2(\theta/2)\cdot 1/a(1-\cos\theta)$$

$$\rho = \frac{-(1+\cot^2(\theta/2))^{3/2}}{\csc^2(\theta/2)}\,2a(1-\cos\theta)$$

$$= -2a \frac{(\operatorname{cosec}^2(\theta/2))^{3/2}}{\operatorname{cosec}^2(\theta/2)} 2\sin^2(\theta/2)$$

$$= -4a \frac{\operatorname{cosec}^3(\theta/2)}{\cos^2 \cancel{\operatorname{ec}}(\theta/2)} \sin^2(\theta/2)$$

$$= -4a \frac{1}{\cancel{\sin(\theta/2)}} \times \cancel{\sin^2}(\theta/2)$$

$$\rho = -4a \sin(\theta/2) \rightarrow (2)$$

$$m_1 m_2 = -1$$

$$(1) \Rightarrow \cot(\theta_1/2) \cdot \cot(\theta_2/2) = -1$$

$$\cot(\theta_1/2) = -1/\cot(\theta_2/2)$$

$$\cot(\theta_1/2) = -\tan(\theta_2/2)$$

$$-\cot(\theta_1/2) = \tan(\theta_2/2)$$

$$\cancel{\tan}(\pi/2 + \theta_1/2) = \cancel{\tan}(\theta_2/2)$$

$$\pi/2 + \theta_1/2 = \theta_2/2$$

$$\pi + \theta_1 = \theta_2$$

$$\rho = -4a \sin(\theta/2)$$

$$\rho_1^2 = 16a^2 \sin^2(\theta_1/2) \rightarrow ③$$

$$\rho_2^2 = 16a^2 \sin^2(\theta_2/2)$$

$$\rho_2^2 = 16a^2 \sin^2\left(\frac{\pi + \theta_1}{2}\right)$$

$$= 16a^2 \sin^2(\pi/2 + \theta_1/2)$$

$$\rho_2^2 = 16a^2 \cos^2(\theta_1/2) \rightarrow ④$$

$\sin(90+\theta) = \cos\theta$

Adding ③ & ④

$$\rho_1^2 + \rho_2^2 = 16a^2 \sin^2(\theta_1/2) + 16a^2\cos^2(\theta_1/2)$$

$$= 16a^2 (\sin^2(\theta_1/2) + \cos^2(\theta_1/2))$$

$$\rho_1^2 + \rho_2^2 = 16a^2$$

Hence they proof

10. S.T. the radius of curvature at any point of the curve. $x = a(\cos t + \log \tan t/2)$ $y = a\sin t$ is inversely proportional to the length of the normal intercepted btw the point on the curve and the axis of x

soln:-

$$x = a(\cos t + \log \tan t/2)$$

$$dx/dt = a\left(-\sin t + \frac{1}{\tan t/2}\right)\sec^2 t/2\,(1/2)$$

$dx/dt = \cancel{\tfrac{a}{2}}\, a\left[-\sin t + \tfrac{1}{2}\right.$ sec $\tfrac{1}{2}$.

$\frac{1}{\frac{\sin t/2}{\cos t/2}}\Big]$

$\cancel{y/x} = \cancel{a \sin t}$

$\cancel{dy/dt = a\cos t}$

$\cancel{dx/dt = a\left[-\sin t + \tfrac{1}{2}\,\frac{\sec^2 t/2}{\sin t/2 \cdot \sec t/2}\right]}$

$\cancel{dx/dt\; a\left[-\sin t + 1/2\right.}$

$$dx/dt = a\left[-\sin t + \tfrac{1}{2}\,\frac{\cos t/2}{\sin t/2}\cdot\frac{1}{\cos^2 t/2}\right]$$

$$dx/dt = a\left[-\sin t + \frac{1}{2\sin t/2\,\cos t/2}\right]$$

$$dx/dt = a\left[-\sin t + 1/\sin t\right]$$

$$= a\left[\frac{-\sin^2 t + 1}{\sin t}\right]$$

$$dx/dt = a\left[\frac{\cos^2 t}{\sin t}\right]$$

$$y = a\sin t$$

$$dy/dt = a\cos t$$

$$dy/dx = \frac{a\cos t}{a\cos^2 t} \times \sin t$$

$$dy/dx = \frac{\sin t}{\cos t}$$

$$dy/dx = \tan t$$

$$d^2y/dx^2 = \sec^2 t \, dt/dx$$

$$d^2y/dx^2 = \sec^2 t \left(\frac{\sin t}{a\cos^2 t}\right)$$

$$d^2y/dx^2 = \frac{\sin t}{a.\cos^4 t}$$

$$\rho = \frac{\left(1 + (dy/dx)^2\right)^{3/2}}{d^2y/dx^2}$$

$$= \frac{\left(1 + (\tan t)^2\right)^{3/2}}{\frac{\sin t}{a\cos^4 t}}$$

$$= \frac{(1 + \tan^2 t)^{3/2}}{\sin t} \times a\cos^4 t$$

$$= \frac{(\sec^2 t)^{3/2}}{\sin t} \times a\cos^4 t$$

$$= \frac{a\sec^3 t}{\sin t} \times \frac{1}{\sec^4 t}$$

$$= \frac{a}{\sin t} \times 1/\sec t$$

$$= \frac{a}{\sin t \sec t}$$

$$= a\frac{\cos t}{\sin t}$$

$$e = a\cot t$$

The length of the normal intercepted btw the point on the curve and the axis of x.

$$= y\sqrt{1+y_1^2}$$

$$= a\sin t\sqrt{1+\tan^2 t}$$

$$= a\sin t\sqrt{\sec^2 t}$$

$$= a\sin t\sec t$$

$$= a\frac{\sin t}{\cos t}$$

$$= a\tan t \Rightarrow \frac{a}{\cot t}$$

$\therefore$ p varies inversely proportional

11. i) Find the radius of curvature at any point $(a\cos\theta, b\sin\theta)$ on the ellipse $x^2/a^2 + y^2/b^2 = 1$.

ii) further S.T. $\rho = \frac{a^2 b^2}{p^3}$ where p is the length of the perpendicular from centre to the tangent at 'θ'.

iii) S.T. the radius of curvature at the end of the major axis is equal to the length of the ~~series~~ latus rectum of the ellipse.

iv) If CP & CD be a pair of conjugate diameters of the ellipse P.T. the radius of curvature at P is CD^3/ab.

v) If ρ_1 & ρ_2 are the radii of curvature the externities of two conjugate semidiameter of an ellipse P.T. $(\rho_1^{2/3} + \rho_2^{2/3})(ab)^{2/3} = a^2 + b^2$

Soln:-

(i) The parametric eqns of the ellipse $x^2/a^2 + y^2/b^2 = 1$. are $x = a\cos\theta$, $y = b\sin\theta$

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}$$

$$x = a\cos\theta$$

$$\frac{dx}{d\theta} = a(-\sin\theta)$$

$$\frac{d^2x}{d\theta^2} = -a\cos\theta$$

$$y = b\sin\theta$$

$$\frac{dy}{d\theta} = b(\cos\theta)$$

$$\frac{d^2y}{d\theta^2} = -b\sin\theta$$

$$\rho = \frac{\left((-a\sin\theta)^2 + (b\cos\theta)^2\right)^{3/2}}{(-a\sin\theta)(-b\sin\theta) - (b\cos\theta)(-a\cos\theta)}$$

$$= \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{a\sin\theta\, b\sin\theta + b\cos\theta\, a\cos\theta}$$

$$= \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab(\sin^2\theta + \cos^2\theta)}$$

$$\rho = \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab} \rightarrow (1)$$

ii) The equ of the tangent at 'θ' is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

The length of the $\perp^r$ from the centre on the tangent is $p = \dfrac{1}{\sqrt{\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2}}}$

$$p = \frac{}{\sqrt{\frac{\cancel{\cos^2\theta}\, a^2\sin^2\theta + b^2\cos^2\theta}{(ab)^2}}}$$

$$p = \frac{ab}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

$$p = \frac{ab}{(a^2\sin^2\theta + b^2\cos^2\theta)^{1/2}}$$

$$p^3 = \frac{(ab)^3}{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}} \rightarrow (2)$$

$$(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2} = \frac{(ab)^3}{p^3}$$

$$\frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab} = \frac{a^2b^2}{p^3}$$

$$\rho = \frac{a^2b^2}{p^3}$$

(iii) The ends of the major axis are A(a,0) & B(-a,0) at the point A, $\theta = 0$ and at point B. $\theta = \pi$ $\boxed{l = 2b^2/a}$

$$\rho(\theta) = \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab}$$

$$= \frac{(a^2 \sin^2\theta + b^2 \cos^2\theta)^{3/2}}{ab}$$

$$= \frac{(a^2(0) + b^2(1))^{3/2}}{ab}$$

$$= \frac{(b^2)^{3/2}}{ab}$$

$$= b^3/ab$$

$$= b^2/a$$

$$= \frac{2b^2}{2a}$$

$$= 1/2 \cdot l$$

14, Let PCP' & DCD' be a pair of conjugate diameters of the ellipse.

let P be the $(a\cos\theta, b\sin\theta)$

the coordinates of D are $[a\cos(\pi/2+\theta), b\sin(\pi/2+\theta)]$

ie, $(-a\sin\theta, b\cos\theta)$.

$\therefore$ Then $CD^2 = a^2\sin^2\theta + b^2\cos^2\theta$

$$\rho = \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab}$$

$$= \frac{(CD^2)^{3/2}}{ab}$$

$$\rho = \frac{CD^3}{ab}$$

v) $$P(\theta) = \rho = \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab}$$

$$ab\rho = (a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}$$

$$(\rho_1)^{2/3}(ab)^{2/3} = a^2\sin^2\theta + b^2\cos^2\theta \rightarrow (1)$$

changing θ into $\pi/2 + \theta$

$$\rho_2 = \frac{(a^2\cos^2\theta + b^2\sin^2\theta)^{3/2}}{ab}$$

$$(\rho_2)^{2/3}(ab)^{2/3} = a^2\cos^2\theta + b^2\sin^2\theta \rightarrow (2)$$

Add eqn (1) & (2)

$$(ab)^{2/3}(\rho_1^{2/3} + \rho_2^{2/3}) = a^2 + b^2.$$

16.10.18
Tuesday

Radius of curvature in polar co-ordinates

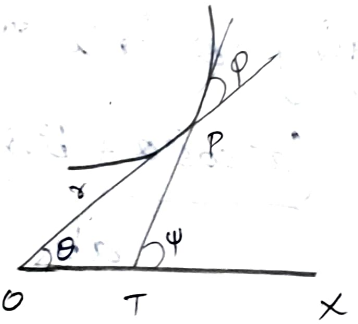
Suppose, the eqn to curve in polar co-ordinates is $r = f(\theta)$

Let O be the pole and OX be the initial line.

Let $P(r, \theta)$, be the polar co-ordinates of the point P on the curve.

Let the tangent at P make with the initial line, at angle ψ.

Let ϕ be the angle b/w the radius OP & PT.



W.K.T : $\psi = \theta + \phi$

$$\frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds} \rightarrow ①$$

W.K.T. $\tan\phi = r \cdot \frac{d\theta}{dr}$

$$\tan\phi = \frac{r}{dr/d\theta} \quad \text{where } dr/d\theta = r_1$$

$$\tan\phi = \frac{r}{r_1}$$

$$\sec^2\varphi \, d\varphi/d\theta = \frac{r_1 \, dr/d\theta - r\left(\frac{d^2r}{d\theta^2}\right)}{r_1^2}$$

$$= \frac{r_1 r_1 - r \cdot r_2}{r_1^2} \quad \text{where } \frac{d^2r}{d\theta^2} = r_2$$

$$\sec^2\varphi \, d\varphi/d\theta = \frac{r_1^2 - r \cdot r_2}{r_1^2}$$

$$\frac{d\varphi}{d\theta} = \frac{r_1^2 - r \cdot r_2}{r_1^2} \cdot \frac{1}{\sec^2\varphi}$$

$$\frac{d\varphi}{d\theta} = \frac{r_1^2 - r \cdot r_2}{r_1^2} \cdot \frac{1}{[1 + \tan^2\varphi]}$$

$$\frac{d\varphi}{d\theta} = \frac{r_1^2 - r \cdot r_2}{r_1^2} \cdot \frac{1}{\left[1 + \frac{r^2}{r_1^2}\right]}$$

$$= \frac{r_1^2 - r \cdot r_2}{r_1^2} \cdot \frac{1}{\left[\frac{r_1^2 + r^2}{r_1^2}\right]}$$

$$\frac{d\varphi}{d\theta} = \frac{r_1^2 - r \cdot r_2}{r_1^2 + r^2} \rightarrow (2)$$

$$\frac{d\varphi}{d\theta} = \frac{d\varphi}{ds} \cdot \frac{ds}{d\theta} = \frac{r_1^2 - r \cdot r_2}{r_1^2 + r^2}$$

$$\frac{d\varphi}{ds} \cdot \frac{ds}{d\theta} = \frac{r_1^2 - r \cdot r_2}{r^2 + r_1^2}$$

$$\frac{d\varphi}{ds} = \frac{r_1^2 - r \cdot r_2}{r^2 + r_1^2} \cdot \frac{d\theta}{ds} \rightarrow (3)$$

Sub eqn (3) in (1)

$$\frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{r_1^2 - r.r_2}{r^2+r_1^2} \cdot d\theta/ds$$

$$= \left[1 + \frac{r_1^2 - r.r_2}{r^2+r_1^2}\right] d\theta/ds$$

$$= \left[\frac{r^2 + r_1^2 + r_1^2 - r.r_2}{r^2+r_1^2}\right] d\theta/ds$$

$$= \left[\frac{r^2 + 2r_1^2 - r.r_2}{r^2+r_1^2}\right] d\theta/ds \rightarrow (4)$$

But $$ds/d\theta = \sqrt{r^2 + (dr/d\theta)^2}$$

$$= \sqrt{r^2 + r_1^2}$$

$$d\theta/ds = \frac{1}{(r^2+r_1^2)^{1/2}} \rightarrow (5)$$

(5) in (4)

$$d\psi/ds = \frac{r^2 + 2r_1^2 - r.r_2}{(r^2+r_1^2)(r^2+r_1^2)^{1/2}}$$

$$d\psi/ds = \frac{r^2 + 2r_1^2 - r.r_2}{(r^2+r_1^2)^{3/2}}$$

$$\frac{ds}{d\psi} = \frac{(r^2+r_1^2)^{3/2}}{r^2 + 2r_1^2 - r.r_2} \rightarrow (6)$$

Note,

suppose the eqn of the curve is

$u = f(\theta)$ where $u = 1/r$

$$r_1 = dr/d\theta = d/d\theta\,(1/u) = -1/u^2\, du/d\theta$$

$$= -u_1/u^2 \rightarrow (1)$$

$$r_2 = d^2r/d\theta^2 = d/d\theta\left(\frac{-u_1}{u_2}\right) = \frac{u^2 . u_2 - u . 2uu_1}{u^4}$$

$$= \frac{2u_1^2}{u^3} - \frac{u_2}{u_1^2} \rightarrow (8)$$

(1) & (8) in eqn (6)

$$\rho = \frac{(u^2 + u_1^2)^{3/2}}{u^3/(u+u_2)}$$

1. Find the radius of curvature at any point on the curve $r^n = a^n \cos n\theta$.

Soln: Given $r^n = a^n \cos n\theta \Rightarrow r^n/a^n = \cos n\theta$

$$\Rightarrow a^n/r^n = \sec n\theta$$

Taking log on both sides

$$\log(r^n) = \log(a^n \cos n\theta)$$

$$n \log r = \log a^n + \log \cos n\theta$$

$$n \log r = n \log a + \log \cos n\theta$$

D.w.r.t. θ

$$n \cdot 1/r \cdot dr/d\theta = 0 - n\tan n\theta$$

$$n/r \cdot dr/d\theta = -n\tan n\theta$$

$$\frac{dr}{d\theta} = -r \tan n\theta \rightarrow (1)$$

$$\frac{d^2r}{d\theta^2} = -\left[r \sec^2 n\theta \cdot n + \frac{dr}{d\theta} \tan n\theta\right]$$

$$= -nr \sec^2 n\theta - \frac{dr}{d\theta} \tan n\theta$$

$$= -nr \sec^2 n\theta + r \tan^2 n\theta$$

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r \cdot r_2}$$

$$= \frac{(r^2 + r^2 \tan^2 n\theta)^{3/2}}{r^2 + 2(r^2 \tan^2 n\theta) - r \cdot [-nr \sec^2 n\theta + r \tan^2 n\theta]}$$

$$= \frac{r^3 (\sec^3 n\theta)}{r^2 + 2r^2 \tan^2 n\theta + nr^2 \sec^2 n\theta - r^2 \tan^2 n\theta}$$

$$= \frac{r^3 \sec^3 n\theta}{r^2 + r^2 \tan^2 n\theta + nr^2 \sec^2 n\theta}$$

$$= \frac{r^3 \sec^3 n\theta}{r^2[(1 + \tan^2 n\theta) + n \sec^2 n\theta]}$$

$$= \frac{r \sec^3 n\theta}{\sec^2 n\theta + n \sec^2 n\theta}$$

$$= \frac{r \sec^3 n\theta}{\sec^2 n\theta (n+1)}$$

$$= \frac{r \sec n\theta}{n+1}$$

$$= \frac{r}{n+1} \cdot \frac{a^n}{r^n} = \frac{a^n}{r^{n-1}(n+1)}$$

18 Tuesday 2. S.T. the radius of curvature at any point on the equiangular spiral. $r = ae^{\theta\cot\alpha}$ is $r\operatorname{cosec}\alpha$

soln:-

Given $r = ae^{\theta\cot\alpha} \longrightarrow (1)$

D.w.r.t. θ,

$$dr/d\theta = ae^{\theta\cot\alpha}\cot\alpha$$

$$dr/d\theta = r\cot\alpha$$

$$\frac{d^2r}{d\theta^2} = ae^{\theta\cot\alpha}\cot\alpha.\cot\alpha$$

$$\frac{d^2r}{d\theta^2} = ae^{\theta\cot\alpha}\cot^2\alpha$$

$$\frac{d^2r}{d\theta^2} = r\cot^2\alpha$$

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-r.r_2}$$

$$= \frac{(r^2+r^2\cot^2\alpha)^{3/2}}{r^2+2r^2\cot^2\alpha-r.r\cot^2\alpha}$$

$$= \frac{(r^2(1+\cot^2\alpha))^{3/2}}{r^2+2r^2\cot^2\alpha-r^2\cot^2\alpha}$$

$$= \frac{r^3\operatorname{cosec}^3\alpha}{r^2+r^2\cot^2\alpha}$$

$$= \frac{r^3\operatorname{cosec}^3\alpha}{r^2(1+\cot^2\alpha)} = \frac{r^3\operatorname{cosec}^3\alpha}{r^2\operatorname{cosec}^2\alpha}$$

$\rho = r \operatorname{cosec} \alpha$

Hence they proof.

3. Find the radius of curvature of the RH, $r^2 = a^2 \sec 2\theta$.

Soln:-

Given $r^2 = a^2 \sec 2\theta \rightarrow (1)$

$$\sec 2\theta = r^2/a^2$$

$$2r\, dr/d\theta = a^2 \sec 2\theta \tan 2\theta \cdot 2$$

$$\cancel{2} r\, dr/d\theta = \cancel{2} a^2 \sec 2\theta \tan 2\theta$$

$$r\, dr/d\theta = r^2 \tan 2\theta$$

$$dr/d\theta = r \tan 2\theta$$

$$\frac{d^2 r}{d\theta^2} = r \sec^2 2\theta \cdot 2 + \tan 2\theta \cdot \frac{dr}{d\theta}$$

$$= 2r\sec^2 2\theta + \tan 2\theta (r \tan 2\theta)$$

$$= 2r\sec^2 2\theta + r\tan^2 2\theta$$

$$\rho = \frac{(r^2 + r^2\tan^2 2\theta)^{3/2}}{r^2 + 2r^2\tan^2 2\theta - r(2r\sec^2 2\theta + r\tan^2 2\theta}$$

$$= \frac{r^3 \sec^3 2\theta}{r^2 + 2r^2 \tan^2 2\theta - 2r^2 \sec^2 2\theta - r^2 \tan^2 2\theta}$$

$$= \frac{r^3 \sec^3 2\theta}{r^2 + r^2 \tan^2 2\theta - 2r^2 \sec^2 2\theta}$$

$$= \frac{r^3 \sec^3 2\theta}{r^2 \sec^2 2\theta - 2r^2 \sec^2 2\theta}$$

$$= \frac{r^3 \sec^3 2\theta}{-r^2 \sec^2 2\theta}$$

$$\rho = -r \sec 2\theta$$

$$\rho = r \sec 2\theta$$

$$\rho = r \cdot r^2/a^2$$

$$\rho = r^3/a^2$$

4. Find the radius of curvature at any point $r^2 = a^2 \cos 2\theta$

Soln :-

Given $r^2 = a^2 \cos 2\theta$

$$2r \, dr/d\theta = a^2(-\sin 2\theta) \cdot 2$$

$$r \, dr/d\theta = -a^2 \sin 2\theta \rightarrow ①$$

again diff. with respect to 'θ'

$$r \cdot d^2r/d\theta^2 + \frac{dr}{d\theta} \cdot \frac{dr}{d\theta^2} = -2a^2 \cos 2\theta$$

$$dr/d\theta = \frac{-a^2 \sin 2\theta}{r}$$

Find the radius of curvature at any point $r^2 = a^2\cos^2 2\theta$

$$r \cdot \frac{d^2r}{d\theta^2} + \left(\frac{dr}{d\theta}\right)^2 = -2r^2$$

$$r \cdot r_2 + r_1^2 = -2r^2$$

$$r \cdot r_2 = -2r^2 - r_1^2 \rightarrow (2)$$

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - \underline{rr_2}}$$

apply this value

$$= \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 + 2r^2 + r_1^2}$$

$$= \frac{(r^2 + r_1^2)^{3/2}}{3r^2 + 3r_1^2}$$

$$= \frac{(r^2 + r_1^2)^{3/2}}{3(r^2 + r_1^2)}$$

$$\rho = \frac{1}{3}(r^2 + r_1^2)^{1/2}$$

$$= \frac{1}{3}\left(r^2 + \frac{a^4\sin^2 2\theta}{r^2}\right)^{1/2}$$

$$= \frac{1}{3}\left(\frac{r^4 + a^4\sin^2 2\theta}{r^2}\right)^{1/2}$$

$$= \frac{1}{3r}\left(a^4\cos^2 2\theta + a^4\sin^2 2\theta\right)^{1/2}$$

$= \frac{1}{3r}\left[a^4(\cos^2 2\theta + \sin^2 2\theta)\right]^{1/2}$

$= \frac{a^2}{3r}\left[\cos^2 2\theta + \sin^2 2\theta\right]^{1/2}$

$\rho = a^2/3r$

2. S.T. the radius of curvature for the centroid $r = a(1+\cos\theta)$ at the point (r, θ) is $\frac{2}{3}\sqrt{2ar}$.

Soln:-

$r = a(1+\cos\theta)$

$\frac{dr}{d\theta} = a(-\sin\theta)$

$dr/d\theta = -a\sin\theta$

$\frac{d^2r}{d\theta^2} = -a\cos\theta$

$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$

$a^2(1+\cos\theta)\cos\theta$

$\rho = \frac{(r^2+a^2\sin^2\theta)^{3/2}}{r^2+2a^2\sin^2\theta+ra\cos\theta}$

$\rho = \frac{(a^2(1+\cos\theta)^2+a^2\sin^2\theta)^{3/2}}{a^2(1+\cos\theta)^2+2a^2\sin^2\theta+a^2(1+\cos\theta)\cos\theta}$

$$\rho = \frac{[a^2(1+\cos\theta)^2 + a^2\sin^2\theta]^{3/2}}{a^2(1+\cos^2\theta+2\cos\theta)+2a^2\sin^2\theta+ar\cos\theta}$$

$$\rho = \frac{([a^2+a^2\cos^2\theta+2a^2\cos\theta]+a^2\sin^2\theta]^{3/2}}{a^2+a^2\cos^2\theta+2a^2\cos\theta+2a^2\sin^2\theta+ar\cos\theta}$$

$$\rho = \frac{(a^2+a^2+2a^2\cos\theta)^{3/2}}{a^2+a^2\cos^2\theta+2a^2\{\cos\theta+2a^2\sin^2\theta+[a\cos\theta[a(1+\cos\theta)]}$$

$$\rho = \frac{(2a^2+2a^2\cos\theta)^{3/2}}{a^2+a^2\cos^2\theta+2a^2\cos\theta+2a^2\sin^2\theta+a^2\cos\theta+a^2\cos^2\theta}$$

$$\rho = \frac{[2a^2]^{3/2}(1+\cos\theta)^{3/2}}{a^2+2a^2\cos^2\theta+3a^2\cos\theta+2a^2\sin^2\theta}$$

$$\rho = \frac{2^{3/2}a^3(1+\cos\theta)^{3/2}}{a^2+2a^2+3a^2\cos\theta}$$

$$\rho = \frac{2^{3/2}a^3(1+\cos\theta)^{3/2}}{3a^2+3a^2\cos\theta}$$

$$= \frac{2^{3/2}a^3(1+\cos\theta)^{3/2}}{3a^2(1+\cos\theta)}$$

$$= \frac{2^{3/2}a^3(1+\cos\theta)^{1/2}}{3a^2}$$

$$= \frac{2\sqrt{2}\, a\,(1+\cos\theta)^{1/2}}{3}$$

$$= \frac{2\sqrt{2}}{3}\sqrt{1+\cos\theta}\,\sqrt{a}\cdot\sqrt{a}$$

$$= \frac{2\sqrt{2}}{3}\sqrt{a}\,\sqrt{a(1+\cos\theta)}$$

$$= \frac{2\sqrt{2}}{3}\sqrt{a}\,\sqrt{r}$$

$$= \frac{2}{3}\sqrt{2}\,\sqrt{a}\,\sqrt{r}$$

$$\rho = \frac{2}{3}\sqrt{2ar}$$

Hence the proof