SUCCESSIVE DIFFERENTIATION AND LEIBNITZ'S THEOREM

1.1 Introduction

Successive Differentiation is the process of differentiating a given function successively n times and the results of such differentiation are called successive derivatives. The higher order differential coefficients are of utmost importance in scientific and engineering applications.

Let $\boldsymbol{f(x)}$ be a differentiable function and let its successive derivatives be denoted by $\boldsymbol{f'(x), f''(x), \ldots, f^{(n)}(x)}$.

Common notations of higher order Derivatives of $\boldsymbol{y = f(x)}$

1st Derivative: $\boldsymbol{f'(x)}$ **or** $\boldsymbol{y'}$ **or** $\boldsymbol{y_1}$ **or** $\boldsymbol{\frac{dy}{dx}}$ **or** $\boldsymbol{Dy}$

2nd Derivative: $\boldsymbol{f''(x)}$ **or** $\boldsymbol{y''}$ **or** $\boldsymbol{y_2}$ **or** $\boldsymbol{\frac{d^2y}{dx^2}}$ **or** $\boldsymbol{D^2y}$

$\vdots$

$\boldsymbol{n^{th}}$ Derivative: $\boldsymbol{f^{(n)}(x)}$ **or** $\boldsymbol{y^{(n)}}$ **or** $\boldsymbol{y_n}$ **or** $\boldsymbol{\frac{d^ny}{dx^n}}$ **or** $\boldsymbol{D^ny}$

1.2 Calculation of n^{th} Derivatives

i. $\boldsymbol{n^{th}}$ Derivative of $\boldsymbol{e^{ax}}$

Let y $= e^{ax}$

$$y_1 = a e^{ax}$$
$$y_2 = a^2 e^{ax}$$
$$\vdots$$
$$y_n = a^n e^{ax}$$

ii. $\boldsymbol{n^{th}}$ Derivative of $\boldsymbol{(ax + b)^m}$, $\boldsymbol{m}$ is a +ve integer greater than $\boldsymbol{n}$

Let y = $(ax + b)^m$

$$y_1 = m\,a(ax + b)^{m-1}$$
$$y_2 = m(m-1)a^2(ax + b)^{m-2}$$
$$\vdots$$
$$y_n = m(m-1)\ldots(m-n+1)a^n(ax + b)^{m-n}$$

$$= \frac{m!}{(m-n)!} a^n (ax + \mathrm{b})^{m-n}$$

iii. $\boldsymbol{n^{th}}$ **Derivative of** $\boldsymbol{y = \log(ax + b)}$

Let $y = \log(ax + b)$

$$y_1 = \frac{a}{(ax+b)}$$

$$y_2 = \frac{-a^2}{(ax+b)^2}$$

$$y_3 = \frac{2!\, a^3}{(ax+b)^3}$$

$$\vdots$$

$$y_n = (-1)^{n-1} \frac{(n-1)!\, a^n}{(ax+b)^n}$$

iv. $\boldsymbol{n^{th}}$ **Derivative of** $\boldsymbol{y = \sin(ax + b)}$

Let $y = \sin(ax + b)$

$$y_1 = a\cos(ax + b) = a\sin\left(ax + b + \frac{\pi}{2}\right)$$

$$y_2 = a^2\cos\left(ax + b + \frac{\pi}{2}\right) = a^2\sin\left(ax + b + \frac{2\pi}{2}\right)$$

$$\vdots$$

$$y_n = a^n\sin\left(ax + b + \frac{n\pi}{2}\right)$$

Similarly if $\boldsymbol{y = \cos(ax + b)}$

$$y_n = a^n\cos\left(ax + b + \frac{n\pi}{2}\right)$$

v. $\boldsymbol{n^{th}}$ **Derivative of** $\boldsymbol{y = e^{ax}\sin(ax + b)}$

Let $y = e^{ax}\sin(bx + c)$

$$y_1 = a\,e^{ax}\sin(bx + c) + e^{ax} b\cos(bx + c)$$

$$= e^{ax}\,[a\sin(bx + c) + b\cos(bx + c)]$$

$$= e^{ax}\,[\, r\, cos\alpha \sin(bx + c) + r\, sin\alpha \cos(bx + c)]$$

Putting $a = r\, cos\alpha,\ b = r\, sin\alpha$

$$= e^{ax}\ r\ \sin(bx + c + \alpha)$$

Similarly $y_2 = e^{ax}\ r^2\ \sin(bx + c + 2\alpha)$

$$\vdots$$

$$y_n = e^{ax}\ r^n\ \sin(bx + c + n\alpha)$$

where $r^2 = a^2 + b^2$ and $\tan\alpha = \frac{b}{a}$

$$\therefore y_n = e^{ax}\ (a^2 + b^2)^{\frac{n}{2}}\ \sin\left(bx + c + n\tan^{-1}\frac{b}{a}\right)$$

Similarly if $\boldsymbol{y = e^{ax}\cos(ax + b)}$

$$y_n = e^{ax}\ r^n\ \cos(bx + c + n\alpha)$$

$$= e^{ax}\ (a^2 + b^2)^{\frac{n}{2}}\ \cos\left(bx + c + n\tan^{-1}\frac{b}{a}\right)$$

Summary of Results

Function	$\boldsymbol{n^{th}}$ **Derivative**
y = e^{ax}	$\boldsymbol{y_n} = a^n e^{ax}$
y = $(ax+b)^m$	$\boldsymbol{y_n} = \begin{cases} \frac{m!}{(m-n)!} a^n (ax+\mathrm{b})^{m-n}, m>0, m> n \\ 0, \quad m>0, \quad m< n, \\ n!\ a^n, \quad m = n \\ \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}, \quad m = -1 \end{cases}$
$y = \log(ax+b)$	$y_n = (-1)^{n-1} \frac{(n-1)!\, a^n}{(ax+b)^n}$
$y = \sin(ax+b)$	$y_n = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$
$y = cos(ax+b)$	$y_n = a^n \cos\left(ax+b+\frac{\mathrm{n}\pi}{2}\right)$
y = $e^{ax} \sin(bx+c)$	$y_n = e^{ax} (a^2+b^2)^{\frac{n}{2}} \sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$
y = $e^{ax} \cos(bx+c)$	$y_n = e^{ax} (a^2+b^2)^{\frac{n}{2}} \cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$

Example 1 Find the n^{th} derivative of $\frac{1}{1-5x+6x^2}$

Solution: Let $y = \frac{1}{1-5x+6x^2}$

Resolving into partial fractions

$$y = \frac{1}{1-5x+6x^2} = \frac{1}{(1-3x)(1-2x)} = \frac{3}{1-3x} - \frac{2}{1-2x}$$

$$\therefore y_n = \frac{3\,(-3)^n(-1)^n n!}{(1-3x)^{n+1}} - \frac{2(-2)^n(-1)^n n!}{(1-2x)^{n+1}}$$

$$\Rightarrow y_n = (-1)^{n+1}\, n! \left[\left(\frac{3}{1-3x}\right)^{n+1} - \left(\frac{2}{1-2x}\right)^{n+1}\right]$$

Example 2 Find the n^{th} derivative of $\sin 6x \cos 4x$

Solution: Let y $= \sin 6x \cos 4x$

$$= \frac{1}{2}(\sin 10\, x + \cos 2\, x)$$

$$\therefore y_n = \frac{1}{2}\left[10^n \sin\left(10x+\frac{n\pi}{2}\right) + 2^n \cos\left(2x+\frac{n\pi}{2}\right)\right]$$

Example 3 Find n^{th} derivative of $sin^2 x cos^3 x$

Solution: Let y = $sin^2 x cos^3 x$

$$= sin^2xcos^2x\ cosx$$
$$= \frac{1}{4} sin^2 2x\ cosx\ = \frac{1}{8}(1 - cos4x)cosx$$
$$= \frac{1}{8}cosx - \frac{1}{8}cos4x\ cosx$$
$$= \frac{1}{8}cosx - \frac{1}{16}(cos3x + cos5x)$$
$$= \frac{1}{16}(2cosx - cos3x - cos5x)$$
$$\therefore y_n = \frac{1}{16}\left[2cos\left(x + \frac{n\pi}{2}\right) - 3^n cos\left(3x + \frac{n\pi}{2}\right) - 5^n cos\left(5x + \frac{n\pi}{2}\right)\right]$$

Example 4 Find the n^{th} derivative of sin^4x

Solution: Let $y = sin^4x\ = (sin^2x)^2$

$$= \left(\frac{1}{2} 2\ sin^2x\right)^2$$
$$= \frac{1}{4}((1 - \cos 2x)^2$$
$$= \frac{1}{4}\left[1 - 2cos2x + \frac{1}{2}(2cos^2 2x)\right]$$
$$= \frac{1}{4}\left[1 - 2cos2x + \frac{1}{2}(1 + cos4x)\right]$$
$$= \frac{3}{8} - \frac{1}{2}cos2x + \frac{1}{8}cos4x$$
$$\therefore y_n = -\frac{1}{2}2^n cos\left(2x + \frac{n\pi}{2}\right) + \frac{1}{8}4^n cos\left(4x + \frac{n\pi}{2}\right)$$

Example 5 Find the n^{th} derivative of $e^{3x}\cos x \sin^2 2x$

Solution: Let $y = e^{3x}\cos x \sin^2 2x$

Now $\cos x \sin^2 2x\ = \frac{1}{2}\ (\cos x - \cos x \cos 4x)$

$$\because \sin^2 2x\ = \frac{1}{2}(1 - \cos 4x)$$
$$= \frac{1}{2}\left(\cos x - \frac{1}{2}(\cos 5x\ + \cos 3x\right)$$
$$\Rightarrow \ y\ = e^{3x}\cos x \sin^2 2x = \frac{1}{2}e^{3x}\cos x - \frac{1}{4}e^{3x}\cos 5x - \frac{1}{4}e^{3x}\cos 3x$$
$$\therefore y_n = \frac{1}{2}e^{3x}\,(9+1)^{\frac{n}{2}}\ \cos\left(x + n\tan^{-1}\frac{1}{3}\right) - \frac{1}{4}e^{3x}\,(9+25)^{\frac{n}{2}}\ \cos\left(5x + n\tan^{-1}\frac{5}{3}\right)$$
$$-\frac{1}{4}e^{3x}\,(9+9)^{\frac{n}{2}}\ \cos\left(3x + n\tan^{-1}\frac{3}{3}\right)$$
$$= \frac{1}{2}e^{3x}\,10^{\frac{n}{2}}\ \cos\left(x + n\tan^{-1}\frac{1}{3}\right) - \frac{1}{4}e^{3x}\,34^{\frac{n}{2}}\ \cos\left(5x + n\tan^{-1}\frac{5}{3}\right)$$
$$-\frac{1}{4}e^{3x}\,18^{\frac{n}{2}}\ \cos(3x + n\tan^{-1} 1)$$

Example 6 If $y = \sin ax + \cos ax$, prove that $y_n = a^n[1 + (-1)^n \sin 2ax\,]^{\frac{1}{2}}$

Solution: $y = \sin ax + \cos ax$

$$\therefore y_n = a^n\left[\sin\left(ax + \frac{n\pi}{2}\right) + \cos\left(ax + \frac{n\pi}{2}\right)\right]$$

$$= a^n\left[\left\{\sin\left(ax + \frac{n\pi}{2}\right) + \cos\left(ax + \frac{n\pi}{2}\right)\right\}^2\right]^{\frac{1}{2}}$$

$$= a^n\left[\sin^2\left(ax + \frac{n\pi}{2}\right) + \cos^2\left(ax + \frac{n\pi}{2}\right) + 2\sin\left(ax + \frac{n\pi}{2}\right).\cos\left(ax + \frac{n\pi}{2}\right)\right]^{\frac{1}{2}}$$

$$= a^n[1 + \sin(2ax + n\pi)]^{\frac{1}{2}}$$

$$= a^n[1 + \sin 2ax \cos n\pi + \cos 2ax \sin n\pi\ ]^{\frac{1}{2}}$$

$$= a^n[1 + (-1)^n \sin 2ax\ ]^{\frac{1}{2}} \quad \because \cos n\pi = (-1)^n \text{ and } \sin n\pi = 0$$

Example 7 Find the n^{th} derivative of $\tan^{-1}\frac{x}{a}$

Solution: Let $y = \tan^{-1}\frac{x}{a}$

$$\Rightarrow y_1 = \frac{dy}{dx} = \frac{1}{a\left(1+\frac{x^2}{a^2}\right)} = \frac{a}{x^2+a^2} = \frac{a}{x^2-(ai)^2}$$

$$= \frac{a}{(x+ai)(x-ai)} = \frac{a}{2ai}\left(\frac{1}{x-ai} - \frac{1}{x+ai}\right)$$

$$= \frac{1}{2i}\left(\frac{1}{x-ai} - \frac{1}{x+ai}\right)$$

Differentiating above $(n-1)$ times w.r.t. x, we get

$$y_n = \frac{1}{2i}\left[\frac{(-1)^{n-1}(n-1)!}{(x-ai)^n} - \frac{(-1)^{n-1}(n-1)!}{(x+ai)^n}\right]$$

Substituting $x = r\,cos\theta,\ a = r\,sin\theta$ such that $\theta = \tan^{-1}\frac{x}{a}$

$$\Rightarrow y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{r^n(\cos\theta - i\,sin\theta)^n} - \frac{1}{r^n(\cos\theta + isin\theta)^n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2ir^n}[(\cos\theta - i\,sin\theta)^{-n} - (\cos\theta + i\,sin\theta)^{-n}]$$

Using De Moivre's theorem, we get

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2ir^n}[\cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta]$$

$$= \frac{(-1)^{n-1}(n-1)!}{r^n}\sin n\theta$$

$$= \frac{(-1)^{n-1}(n-1)!}{\left(\frac{a}{\sin\theta}\right)^n}\sin n\theta \qquad \because a = r\,sin\theta$$

$$= \frac{(-1)^{n-1}(n-1)!}{a^n}\sin n\theta\ \sin^n\theta \qquad \text{where } \theta = \tan^{-1}\frac{a}{x}$$

Example 8 Find the n^{th} derivative of $\frac{1}{1+x+x^2}$

Solution: Let $y = \frac{1}{1+x+x^2}$

$$= \frac{1}{(x-w)(x-w^2)} \quad \text{where } w = \frac{-1+i\sqrt{3}}{2} \text{ and } w^2 = \frac{-1-i\sqrt{3}}{2}$$

Resolving into partial fractions

$$y = \frac{1}{w-w^2}\left(\frac{1}{x-w} - \frac{1}{x-w^2}\right)$$

$$= \frac{1}{i\sqrt{3}}\left(\frac{1}{x-w} - \frac{1}{x-w^2}\right) = \frac{-i}{\sqrt{3}}\left(\frac{1}{x-w} - \frac{1}{x-w^2}\right)$$

Differentiating n times w.r.t. , we get

$$y_n = \frac{-i}{\sqrt{3}}\left[\frac{(-1)^n n!}{(x-w)^{n+1}} - \frac{(-1)^n n!}{(x-w^2)^{n+1}}\right]$$
$$= \frac{-i\,(-1)^n n!}{\sqrt{3}}\left[\frac{1}{(x-w)^{n+1}} - \frac{1}{(x-w^2)^{n+1}}\right]$$
$$= \frac{i\,(-1)^{n+1} n!}{\sqrt{3}}\left[\frac{1}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^{n+1}} - \frac{1}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{n+1}}\right]$$
$$= \frac{i\,2^{n+1}\,(-1)^{n+1} n!}{\sqrt{3}}\left[\frac{1}{\left(2x+1-i\sqrt{3}\right)^{n+1}} - \frac{1}{\left(2x+1+i\sqrt{3}\right)^{n+1}}\right]$$

Substituting $2x+1 = r\,cos\theta,\ \sqrt{3} = r\,sin\theta$ such that $\theta = \tan^{-1}\frac{\sqrt{3}}{2x+1}$

$$y_n = \frac{i\,2^{n+1}\,(-1)^{n+1} n!}{\sqrt{3}\ r^{n+1}}\left[(cos\theta - isin\theta)^{-(n+1)} - (cos\theta + isin\theta)^{-(n+1)}\right]$$

Using De Moivre's theorem, we get

$$y_n = \frac{i\,2^{n+1}\,(-1)^{n+1} n!}{\sqrt{3}\left(\frac{\sqrt{3}}{sin\theta}\right)^{n+1}}\ [cos(n+1)\theta + i\,sin(n+1)\theta - cos(n+1)\theta + isin(n+1)\theta]$$

$$\because \sqrt{3} = r\,sin\theta$$

$$= \frac{i\,2^{n+1}\,(-1)^{n+1} n!}{\left(\sqrt{3}\right)^{n+2}}\ 2i\,sin(n+1)\theta\,sin^{n+1}\theta$$

$$= \frac{(-2)^{n+2} n!}{\sqrt{3}^{n+2}}\ sin(n+1)\theta\,sin^{n+1}\theta \quad \text{where } \theta = \tan^{-1}\frac{\sqrt{3}}{2x+1}$$

Example 9 If $y = x + \tan x$, show that $\cos^2 x\,\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{d}x^2} - 2y + 2x = 0$

Solution: $y = x + \tan x$

$$\Rightarrow \frac{\mathrm{dy}}{\mathrm{d}x} = 1 + sec^2 x$$

$$\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{d}x^2} = 2\sec x\,(\sec x \tan x) = 2\ sec^2 x \tan x$$

$$\therefore\ \cos^2 x\,\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{d}x^2} - 2y + 2x = 2\cos^2 x\,sec^2 x \tan x - 2(x + \tan x) + 2x$$
$$= 2\tan x - 2x - 2\tan x + 2x$$
$$= 0$$

Example 10 If $y = \log\left(x + \sqrt{x^2+1}\right)$, show that $(1+x^2)\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{d}x^2} + x\frac{dy}{dx} = 0$

Solution: $y = \log\left(x + \sqrt{x^2+1}\right)$

$$\Rightarrow \frac{\mathrm{dy}}{\mathrm{d}x} = \frac{1+\frac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow (\sqrt{1+x^2})\frac{\mathrm{dy}}{\mathrm{d}x} = 1$$

Differentiating both sides w.r.t. x , we get

$$(\sqrt{1+x^2})\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{d}x^2} + \frac{x}{\sqrt{1+x^2}}\frac{\mathrm{dy}}{\mathrm{d}x} = 0$$

$$\Rightarrow (1+x^2)\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{d}x^2} + x\frac{dy}{dx} = 0$$

Exercise 1 A

1. Find the n^{th} derivative of $\frac{x^4}{(x-1)(x-2)}$

   Ans. $(-1)^n n!\left[\frac{16}{(\mathrm{x}-2)^{n+1}} - \frac{1}{(\mathrm{x}-1)^{n+1}}\right]$

2. Find the n^{th} derivative of $\cos x \cos 2x \cos 3x$

   Ans. $\frac{1}{4}\left[2^n \cos\left(2x + \frac{n\pi}{2}\right) + 4^n \cos\left(4x + \frac{n\pi}{2}\right) + 6^n \cos\left(6x + \frac{n\pi}{2}\right)\right]$

3. If $x = \sin t$, $y = \sin at$, show that $(1-x^2)\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{d}x^2} - x\frac{dy}{dx} + a^2y = 0$

4. If $p^2 = a^2 cos^2\theta + b^2 sin^2\theta$, show that $p + \frac{\mathrm{d}^2p}{\mathrm{d}\theta^2} = \frac{a^2b^2}{p^3}$

5. If $y = \frac{x}{x^2+a^2}$, find y_n i.e. the n^{th} derivative of y

   Ans. $\frac{(-1)^n n!}{a^{n+1}} cos(n+1)\theta\ sin^{n+1}\theta$ where $\theta = \tan^{-1}\frac{x}{a}$

6. If $y = e^x sin^2 x$, find y_n i.e. the n^{th} derivative of y

   Ans. $\frac{1}{2}\ e^x\left[1 - 16\left(2\mathrm{x} + \frac{n\pi}{2}\right)\right]$

7. Find n^{th} differential coefficient of $y = \log[(ax+b)(cx+d)]$

   Ans. $y_n = (-1)^{n-1}(n-1)!\left[\frac{a^n}{(ax+b)^n} + \frac{c^n}{(cx+d)^n}\right]$

8. If y = $x\log\frac{x-1}{x+1}$, show that $y_n = (-1)^{n-1}(n-2)!\left[\frac{x-n}{(x-1)^n} + \frac{x+n}{(x+1)^n}\right]$

9. If $y = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$, show that $y_n = \frac{1}{2}(-1)^{n-1}(n-1)!\sin n\theta\ sin^n\theta$

1.2 LEIBNITZ'S THEOREM

If u and v are functions of x such that their n^{th}derivatives exist, then the n^{th}derivative of their product is given by

$$(u\,v)_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots + n_{C_r} u_{n-r} v_r + \cdots + u v_n$$

where u_r and v_r represent r^{th}derivatives of u and v respectively.

Example11 Find the n^{th} derivative of $x \log x$

Solution: Let $u = \log x$ and $v = x$

Then $u_n = (-1)^{n-1} \frac{(n-1)!}{x^n}$ and $u_{n-1} = (-1)^{n-2} \frac{(n-2)!}{x^{n-1}}$

By Leibnitz's theorem, we have

$$(u\,v)_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots + n_{C_r} u_{n-r} v_r + \cdots + u v_n$$

$$\Rightarrow (x \log x)_n = (-1)^{n-1} \frac{(n-1)!}{x^n} x + n(-1)^{n-2} \frac{(n-2)!}{x^{n-1}} + 0$$

$$= (-1)^{n-1} \frac{(n-1)!}{x^{n-1}} + n(-1)^{n-2} \frac{(n-2)!}{x^{n-1}}$$

$$= (-1)^{n-2} \frac{(n-2)!}{x^{n-1}} [-(n-1) + n]$$

$$= (-1)^{n-2} \frac{(n-2)!}{x^{n-1}}$$

Example 12 Find the n^{th} derivative of $x^2 e^{3x} \sin 4x$

Solution: Let $u = e^{3x} \sin 4x$ and $v = x^2$

Then $u_n = e^{3x}\, 25^{\frac{n}{2}} \sin\left(4x + n \tan^{-1}\frac{4}{3}\right)$

$$= e^{3x}\, 5^n \sin\left(4x + n \tan^{-1}\frac{4}{3}\right)$$

By Leibnitz's theorem, we have

$$(u\,v)_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots + n_{C_r} u_{n-r} v_r + \cdots + u v_n$$

$$\Rightarrow \left(x^2 e^{3x} \sin 4x\right)_n = x^2 e^{3x}\, 5^n \sin\left(4x + n \tan^{-1}\frac{4}{3}\right) +$$

$$2nxe^{3x}\, 5^{n-1} \sin\left(4x + (n-1) \tan^{-1}\frac{4}{3}\right) +$$

$$n(n-1)e^{3x}\, 5^{n-2} \sin\left(4x + (n-2) \tan^{-1}\frac{4}{3}\right) + 0$$

$$= e^{3x}5^n \left[x^2 \sin\left(4x + n\tan^{-1}\frac{4}{3}\right) + \frac{2nx}{5}\sin\left(4x + (n-1)\tan^{-1}\frac{4}{3}\right) + \frac{n(n-1)}{25}\sin\left(4x + (n-2)\tan^{-1}\frac{4}{3}\right)\right]$$

Example 13 If $y = a\cos(\log x) + b\,sin(\log x)$, show that

$$x^2y_{n+2} + (2n+1)xy_{n+1} + n(n+1)y_n = 0$$

Solution: Here $y = a\cos(\log x) + b\sin(\log x)$

$$\Rightarrow y_1 = \frac{-a}{x}\sin(\log x) + \frac{b}{x}\cos(logx)$$

$$\Rightarrow xy_1 = -a\sin(logx) + b\,cos(logx)$$

Differentiating both sides w.r.t. x , we get

$$xy_2 + y_1 = -\frac{a}{x}\cos(logx) + \frac{-b}{x}\ sin(logx)$$

$$\Rightarrow x^2y_2 + xy_1 = -\{a\cos(logx) + b\ sin(logx)\}$$

$$= -y$$

$$\Rightarrow x^2y_2 + xy_1 + y = 0$$

Using Leibnitz's theorem, we get

$$(y_{n+2}x^2 + n_{C_1}y_{n+1}2x + n_{C_2}y_n.2) + \left(y_{n+1}x + n_{C_1}y_n.1\right) + y_n = 0$$

$$\Rightarrow y_{n+2}x^2 + y_{n+1}2nx + n(n-1)y_n + y_{n+1}x + ny_n + y_n = 0$$

$$\Rightarrow x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$$

Example 14 If $y = \log(x + \sqrt{1+x^2})$

Prove that $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$

Solution: $y = \log(x + \sqrt{1+x^2}$

$$\Rightarrow y_1 = \frac{1}{x+\sqrt{1+x^2}}\left(1 + \frac{1}{2\sqrt{1+x^2}}2x\right) = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow (1+x^2){y_1}^2 = 1$$

Differentiating both sides w.r.t. x, we get

$$(1+x^2)2y_1y_2 + 2x{y_1}^2 = 0$$

$$\Rightarrow (1+x^2)y_2 + xy_1 = 0$$

Using Leibnitz's theorem

$$[y_{n+2}(1+x^2)+n_{C_1}y_{n+1}2x+n_{C_2}y_n.2]+\left(y_{n+1}x+n_{C_1}y_n.1\right)=0$$

$$\Rightarrow y_{n+2}(1+x^2)+y_{n+1}2nx+n(n-1)y_n+y_{n+1}x+ny_n=0$$

$$\Rightarrow (1+x^2)y_{n+2}+(2n+1)xy_{n+1}+n^2y_n=0$$

Example 15 If $y=\sin\,(m\,sin^{-1}x)$, show that

$(1-x^2)y_{n+2}=(2n+1)xy_{n+1}+(n^2-m^2)y_n$. Also find $y_n(0)$

Solution: Here $y=\sin(m\,sin^{-1}x)$ ……①

$$\Rightarrow y_1=\frac{m}{\sqrt{1-x^2}}\cos(m\,sin^{-1}x) \quad \text{……②}$$

$$\Rightarrow (1-x^2){y_1}^2=\mathrm{m}^2\cos^2(m\,sin^{-1}x)$$

$$\Rightarrow (1-x^2){y_1}^2=\mathrm{m}^2[1-\sin^2(m\,sin^{-1}x)]$$

$$\Rightarrow (1-x^2){y_1}^2=m^2(1-y^2)\text{……③}$$

$$\Rightarrow (1\text{-}x^2){y_1}^2+m^2y^2=m^2$$

Differentiating w.r.t. x, we get

$$(1-x^2)2y_1y_2+{y_1}^2(-2x)+m^22yy_1=0$$

$$\Rightarrow (1-x^2)y_2-xy_1+m^2y=0$$

Using Leibnitz's theorem, we get

$$[y_{n+2}(1-x^2)+n_{C_1}y_{n+1}(-2x)+n_{C_2}y_n(-2)]-\left(y_{n+1}x+n_{C_1}y_n1\right)+m^2y_n=0$$

$$\Rightarrow y_{n+2}(1-x^2)-y_{n+1}2nx-n(n-1)y_n-(y_{n+1}x+ny_n)+m^2y_n=0$$

$$\Rightarrow (1-x^2)y_{n+2}=(2n+1)xy_{n+1}+(n^2-m^2)y_n\text{……④}$$

Putting $x=0$ in ①, ②and ③

$y(0)=0, y_1(0)=m$ and $y_2(0)=0$

Putting $x=0$ in ④

$y_{n+2}(0)=(n^2-m^2)y_n(0)$

Putting $n=1,2,3 \ldots \ldots$ in the above equation, we get

$y_3(0)=(1^2-m^2)y_1(0)$

$=(1^2-m^2)m \quad \because y_1(0)=m$

$$y_4(0) = (2^2 - m^2)y_2(0)$$

$$= 0 \qquad \because y_2(0) = 0$$

$$y_5(0) = \left(3^2 - m^2\right)y_3(0)$$

$$= m(1^2 - m^2)(3^2 - m^2)$$

$$\vdots$$

$$\Rightarrow \quad y_n(0) = \begin{cases} 0\,, & if\ n\ is\ even \\ m(1^2 - m^2)(3^2 - m^2)\ldots[(n-2)^2) - m^2]\,, & if\ n\ is\ odd \end{cases}$$

Example 16 If $y = e^{msin^{-1}x}$, show that

$(1 - x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 + m^2)y_n = 0$. Also find $y_n(0)$.

Solution: Here $y = e^{msin^{-1}x}$ …①

$$\Rightarrow y_1 = \frac{m}{\sqrt{1-x^2}}\, e^{m\, sin^{-1}x}$$

$$= \frac{my}{\sqrt{1-x^2}} \quad \ldots\ldots②$$

$$\Rightarrow (1 - x^2){y_1}^2 = m^2y^2$$

Differentiating above equation w.r.t. x , we get

$$(1 - x^2)2y_1y_2 + {y_1}^2(-2x) = m^2 2yy_1$$

$$\Rightarrow (1 - x^2)y_2 - xy_1 - m^2y = 0 \qquad \ldots\ldots③$$

Differentiating above equation n times w.r.t. x using Leibnitz's theorem, we get

$$[y_{n+2}(1 - x^2) + n_{C_1}y_{n+1}(-2x) + n_{C_2}y_n(-2)] - \left(y_{n+1}x + n_{C_1}y_n1\right) - m^2y_n = 0$$

$$\Rightarrow y_{n+2}(1 - x^2) - y_{n+1}2nx - n(n-1)y_n - (y_{n+1}x + ny_n) - m^2y_n = 0$$

$$\Rightarrow (1 - x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 + m^2)y_n = 0 \ldots\ldots④$$

To find $y_n(0)$: Putting $x = 0$ in ①, ② and ③

$y(0) = 1$, $y_1(0) = m$ and $y_2(0) = m^2$

Also putting $x = 0$ in , we get

$$y_{n+2}(0) = (n^2 + m^2)y_n(0)$$

Putting $n = 1,2,3 \ldots$ in the above equation, we get

$$y_3(0) = (1^2 + m^2)y_1(0)$$

$$=(1^2 + m^2)m \qquad \because y_1(0) = m$$

$$y_4(0) = (2^2 + m^2)y_2(0)$$

$$= m^2(2^2 + m^2) \qquad \because y_2(0) = m^2$$

$$y_5(0) = \left(3^2 + m^2\right)y_3(0)$$

$$= m(1^2 + m^2)(3^2 + m^2)$$

$$\vdots$$

$$\Rightarrow \quad y_n(0) = \begin{cases} m^2(2^2 + m^2) \ldots [(n-2)^2 + m^2] \ , & if\ n\ is\ even \\ m(1^2 + m^2)(3^2 + m^2) \ldots [(n-2)^2 + m^2] \, , & if\ n\ is\ odd \end{cases}$$

Example 17 If $y = tan^{-1}x$, show that

$(1 - x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$. Also find $y_n(0)$

Solution: Here $y = tan^{-1}x$.....①

$$\Rightarrow y_1 = \frac{1}{1+x^2} \quad \ldots\ldots②$$

$$y_2 = \frac{-2x}{1+x^2}$$

$$\Rightarrow (1 + x^2)y_2 + 2xy_1 = 0 \ \ldots\ldots③$$

Differentiating equation ③ n times w.r.t. x using Leibnitz's theorem

$$[y_{n+2}(1 + x^2) + n_{C_1}y_{n+1}(2x) + n_{C_2}y_n(2)] + 2\left(y_{n+1}x + n_{C_1}y_n 1\right) = 0$$

$$\Rightarrow y_{n+2}(1 + x^2) + y_{n+1}2nx + n(n-1)y_n + 2(y_{n+1}x + ny_n) = 0$$

$$\Rightarrow (1 + x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0 \ldots\ldots④$$

To find $y_n(0)$: Putting $x = 0$ in ①, ②and ③, we get

$y(0) = 0$, $y_1(0) = 1$ and $y_2(0) = 0$

Also putting $x = 0$ in ④, we get

$$y_{n+2}(0) = -n(n+1)y_n(0)$$

Putting $n = 1,2,3$ ... in the above equation, we get

$y_3(0) = -1(2)y_1(0)$

$= -2 \qquad \because y_1(0) = 1$

$y_4(0) = -2(3)y_2(0)$

$= \; 0 \qquad \because y_2(0) = 0$

$y_5(0) = -3(4)y_3(0)$

$= -3(4)(-2) = 4!$

$y_6(0) = -4(5)y_4(0) = 0$

$y_7(0) = -5(6)y_5(0) = -5(6)4! = -(6!)$

$\vdots$

$\Rightarrow \quad y_{2n+1}(0) = (-1)^n(2n)!$ and $\quad y_{2n}(0) = 0$

Example18 If $y = (sin^{-1}x)^2$, show that $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - n^2 y_n = 0$. Also find $y_n(0)$

Solution: Here $y = (sin^{-1}x)^2$.....①

$\Rightarrow y_1 = 2sin^{-1}x.\frac{1}{\sqrt{1-x^2}}$ ......②

Squaring both the sides, we get

$(1 - x^2){y_1}^2 = 4\,(sin^{-1}x)^2$

$\Rightarrow (1 - x^2){y_1}^2 = 4\,(y)^2$

Differentiating the above equation w.r.t. x, we get

$(1 - x^2)2y_1y_2 + {y_1}^2(-2x) - 4y_1 = 0$

$\Rightarrow (1 - x^2)y_2 + y_1(-x) - 2 = 0$ ......③

Differentiating the above equation n times w.r.t. x using Leibnitz's theorem, we get

$[y_{n+2}(1 - x^2) + n_{C_1}y_{n+1}(-2x) + n_{C_2}y_n(-2)] - (y_{n+1}x + n_{C_1}y_n 1) = 0$

$\Rightarrow y_{n+2}(1 - x^2) - y_{n+1}2nx - n(n-1)y_n - (y_{n+1}x + ny_n) = 0$

$\Rightarrow (1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - y_n n^2 = 0$......④

To find $y_n(0)$: Putting $x = 0$ in ①, ②and ③, we get

$y(0) = 0\,, y_1(0) = 0$ and $y_2(0) = 2$

Also putting $x = 0$ in ④, we get

$$y_{n+2}(0) = n^2 y_n(0)$$

Putting $n = 1,2,3 \ldots$ in the above equation, we get

$$y_3(0) = 1^2 y_1(0)$$

$$= 0 \qquad \because y_1(0) = 0$$

$$y_4(0) = 2^2 y_2(0)$$

$$= 2^2 2 \qquad \because y_2(0) = 2$$

$$y_5(0) = 3^2 y_3(0) = 0$$

$$y_6(0) = 4^2 y_4(0) = 4^2 2^2 2$$

$$\vdots$$

$$\Rightarrow \quad y_n(0) = \begin{cases} 0\,, & if\ n\ is\ odd \\ 2.\,2^2.\,4^2 \ldots\ldots\ldots(n-2)^2\,, & if\ n\ is\ even \end{cases}$$

Exercise 1 B

1 .Find y_n , if $y = x^3 \cos x$

Ans.$x^3 cos\left(x + \frac{n\pi}{2}\right) + 3nx^2 cos\left[x + \frac{1}{2}(n-1)\pi\right] + 3n(n-1)xcos\left[x + \frac{1}{2}(n-2)\pi\right] + n(n-1)(n-2)cos\left[x + \frac{1}{2}(n-3)\pi\right]$

2. Find y_n , if $y = x^2 e^x \cos x$

Ans. $2^{\frac{n}{2}} e^x cos\left(x + \frac{n\pi}{4}\right)$

3. If $y^{\frac{1}{m}} + y^{\frac{-1}{m}} = 2x$, prove that $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$

4. If $y\sqrt{1+x^2} = \log\left(x + \sqrt{1+x^2}\right)$, prove that

$$(1+x^2)y_{n+2} + (2n+3)xy_{n+1} + (n+1)^2 y_n = 0$$

5. If $y = \left[x + \sqrt{1+x^2}\right]^m$, prove that $(x^2+1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$

6 If $y = (sinh^{-1}x)^2$, show that

$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0$. Also find $y_n(0)$.

Ans. $y_{2n+1}(0) = 0$ and $y_{2n}(0) = (-1)^{n-1} 2.\,2^2.\,4^2 \ldots\ldots\ldots(2n-2)^2$

7. If $y = \cos\left(m\ sin^{-1}x\right)$, show that

$(1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2-m^2)y_n$. Also find $y_n(0)$.

8. If $f(x) = \tan x$, prove that $f^{n}(0) - n_{C_2} f^{n-2}(0) + n_{C_4} f^{n-4}(0) - \cdots \ = sin \frac{n\pi}{2}$