UNIT - IV

Infinite series:

Let $(a_n) = a_1, a_2 \cdots a_n \cdots$ be a sequence of real number than

Cauchy sequence in R and hence

(s_n) is convergent.

The given series convergest.

Theorem: 7

Harmonic series $\sum \frac{1}{n^p}$ $p>1$ and diverges if $p \leq 1$.

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proof:

Case (i)

Let $p=1$

Then the series $\sum (1/n)$ which diverges.

Case (ii)

Let $p<1$

Then $n^p < n \ \forall n$

$$\therefore \frac{1}{n^p} > 1/n \ \forall n.$$

By comparison test $\sum 1/n^p$ diverges.

Case (iii)

Let $p>1$

Let $S_n = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots \frac{1}{n^p}$

$$S_{2^{n+1}-1} = 1 + \frac{1}{2^p} + \cdots + \frac{1}{(2^{n+1}-1)^p}$$

$$= 1 + \left(\frac{1}{2^p} + \frac{1}{3}\right.$$

a_n 2^n

Rabies test:

RAABE'S TEST:-

Theorem:-

Let us compare the series Σu_n with the series $\Sigma \frac{1}{n^p}$. $\Sigma \frac{1}{n^p}$ is convergent when $p > 1$ and divergent when $p < 1$.

Proof:-

For convergent,

Let $$u_n = \frac{1}{n^p}$$

$$u_{n+1} = \frac{1}{(n+1)^p}$$

$$\frac{u_{n+1}}{u_n} < \frac{n^p}{(n+1)^p}$$

i.e) $\frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p}$

$$> \frac{n^p(1+1/n)^p}{n^p}$$

$$\frac{u_n}{u_{n+1}} > (1+1/n)^p$$

We know that,

$$(1+1/x)^n = 1 + \frac{n}{x} + \frac{n(n-1)}{2!}\frac{1}{x^2} + \cdots$$

here,

$$\frac{u_n}{u_{n+1}} > 1 + \frac{p}{n} + \frac{p(p-1)}{2!}\frac{1}{n^2} + \cdots$$

$$\frac{u_n}{u_{n+1}} > 1 + \frac{p}{n} + \frac{p(p-1)}{2!}\frac{1}{n^2}$$

$$\frac{u_n}{u_{n+1}} - 1 > \frac{p}{n} + \frac{p(p-1)}{2!}\frac{1}{n^2}$$

$$\frac{u_n}{u_{n+1}} - 1 > \frac{1}{n}\left[p + \frac{p(p-1)}{2!}\frac{1}{n}\right]$$

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) > p + \frac{p(p-1)}{2!}\frac{1}{n}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) > \lim_{n\to\infty}\left[p + \frac{p(p-1)}{2!}\frac{1}{n}\right]$$

$$> p$$

$\therefore \Sigma u_n$ is convergent, when $p > 1$.

For divergent,

Let $\frac{u_{n+1}}{\ldots} > \frac{n^p}{\ldots^p}$

i.e) $\dfrac{u_n}{u_{n+1}} < \dfrac{(n+1)^p}{n^p}$

$$< \frac{n^p(1+1/n)^p}{n^p}$$

$$< (1+1/n)^p$$

$$\frac{u_n}{u_{n+1}} < 1+\frac{p}{n}+\frac{p(p-1)}{2!}\frac{1}{n^2}+\cdots$$

$$< 1+\frac{p}{n}+\frac{p(p-1)}{2!}\frac{1}{n^2}$$

$$\frac{u_n}{u_{n+1}}-1 < \frac{p}{n}+\frac{p(p-1)}{2!}\frac{1}{n^2}$$

$$\frac{u_n}{u_{n+1}}-1 < \frac{1}{n}\left(p+\frac{p(p-1)}{2!}\frac{1}{n}\right)$$

$$n\left(\frac{u_n}{u_{n+1}}-1\right) < \left(p+\frac{p(p-1)}{2!}\frac{1}{n}\right)$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) < \lim_{n\to\infty}\left(p+\frac{p(p-1)}{2!}\frac{1}{n}\right)$$

$$< p$$

$\therefore$ Σu_n is divergent, when $p<1$.

Corollary :-

The Series whose general terms u_n is convergent (or) divergent according as $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) > 1$ (or) $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) < 1$. This is known as Raabe's test.

1 Test for convergence and divergence of the series $\frac{x}{1\cdot 2}+\frac{x^2}{2\cdot 3}+\frac{x^3}{3\cdot 4}+\cdots$

Soln:-

$$u_n=\frac{x^n}{n(n+1)}$$

$$u_{n+1}=\frac{x^{n+1}}{(n+1)(n+2)}$$

$$\frac{u_{n+1}}{u_n}=\frac{x^{n+1}}{(n+1)(n+2)}\times\frac{n(n+1)}{x^n}$$

$$=\frac{nx}{n+2}$$

$$=\frac{x}{(1+2/n)}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{x}{(1+2/n)}$$

$$=x$$

By D'Alembert's ratio test

Σu_n is convergent, if $x<1$

Σu_n is divergent, if $x>1$.

The test is failed if $x=1$

Using comparison test,

put $x=1$.

$$\therefore\ \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots$$

$$u_n=\frac{1}{n(n+1)}$$

$$u_n = \frac{1}{n^2(1+1/n)}$$

Let, $V_n = \frac{1}{n^2}$

$$\therefore \frac{u_n}{v_n} = \frac{1}{n^2(1+1/n)} \times \frac{n^2}{1}$$

$$= \frac{1}{(1+1/n)}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{(1+1/n)}$$

$$= 1 \neq 0 \text{ finite}$$

$\therefore \Sigma V_n$ is divergent series. Σu_n is also divergent series.

Examine the convergence of the series $\sum_{1}^{\infty} \left(\frac{n}{n+1}\right)^2 x^n$.

Soln:-

$$a_n = \left(\frac{n}{n+1}\right)^2 x^n$$

$$a_{n+1} = \left(\frac{n+1}{n+2}\right)^2 x^{n+1}$$

$$\frac{a_{n+1}}{a_n} = \left(\frac{n+1}{n+2}\right)^2 x^{n+1} \times \left(\frac{n+1}{n}\right)^2 \left(\frac{1}{x^n}\right)$$

$$= \frac{(1+1/n)^2}{(1+2/n)^2} (x) (1+1/n)^2$$

$$\frac{a_{n+1}}{a_n} = \frac{(1+1/n)^4 x}{(1+2/n)^2}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(1+1/n)^4 x}{(1+2/n)^2}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = x$$

By D'Alembert's ratio test

Σa_n is convergent, if $x < 1$

Σa_n is divergent, if $x > 1$.

Test is failed if $x = 1$.

Using comparison test,

put $x = 1$.

$$a_n = \left(\frac{n}{n+1}\right)^2$$

$$a_n = \frac{1}{(1+1/n)^2}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{(1+1/n)^2}$$

$$= 1 \neq 0.$$

$\therefore$ Σa_n is divergent series.

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3) Examine the convergence of the series

$$\frac{1}{3}x + \frac{1\cdot 2}{3\cdot 5}x^2 + \frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}x^3 + \cdots$$

Soln:-

$$a_n = \frac{(1\cdot 2\cdot 3 \cdots n)x^n}{[3\cdot 5\cdot 7 \cdots (2n+1)]}$$

$$a_{n+1} = \frac{[1\cdot 2\cdot 3 \cdots n(n+1)]x^{n+1}}{[3\cdot 5\cdot 7 \cdots (2n+1)(2n+3)]}$$

$$\frac{a_{n+1}}{a_n} = \frac{[1\cdot 2\cdot 3 \cdots n(n+1)]\, x^{n+1}}{[3\cdot 5\cdot 7 \cdots (2n+1)(2n+3)]} \times \frac{3\cdot 5 \cdots (2n+1)}{(1\cdot 2\cdot 3 \cdots n)\, x^n}$$

$$= \frac{x(n+1)}{(2n+3)}$$

$$= \frac{nx(1+1/n)}{n(2+3/n)}$$

$$\frac{a_{n+1}}{a_n} = \frac{x(1+1/n)}{(2+3/n)}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x(1+1/n)}{(2+3/n)}$$

$$= \frac{x}{2}$$

By the D'Alembert's ratio test

Σa_n is convergent, if $x < 2$.

Σa_n is divergent, if $x > 2$.

Test is failed if $x = 2$.

Using the comparison test,

put $x = 2$.

$$\frac{1}{3}(2) + \frac{1\cdot 2}{3\cdot 5}(2^2) + \frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}(2^3) + \cdots$$

$$a_n = \frac{(1\cdot 2\cdot 3 \cdots n)\, 2^n}{[3\cdot 5\cdot 7 \cdots (2n+1)]}$$

$$= \frac{(n!)\, 2^n}{(2n+1)!}$$

$$= \frac{n(n-1)!\, 2^n}{(2n+1)(2n-1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{n!}{n^n} \times \frac{(n+1)^{n+1}}{(n+1)!}$$

$$= \frac{n!\,(n+1)^n\,(n+1)}{n^n\,(n+1)\,n!}$$

$$= \frac{(n+1)^n}{n^n}$$

$$= (1+1/n)^n$$

$$= e$$

$$= 2.718 < 1.$$

$\therefore$ The series is convergent.

Test the convergence of the series

$$\sum \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$$

Soln:-

$$u_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$$

$$(u_n)^{1/n} = \left[\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}\right]^{1/n}$$

$$= \left(1+\frac{1}{\sqrt{n}}\right)^{-\frac{n^{3/2}}{n}}$$

$$= \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2-1}}$$

$$= \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{1/2}}$$

$$= \left(1+\frac{1}{\sqrt{n}}\right)^{-\sqrt{n}}$$

$$= \frac{1}{(1+1/\sqrt{n})^{\sqrt{n}}}$$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{1}{(1+1/\sqrt{n})^{\sqrt{n}}}$$

$$= \frac{1}{e}$$

$= \frac{1}{2.718}$

$= 0.367 < 1$

The series is convergent

Conditionally convergence series

The series Σu_n containing +ve and -ve term to be conditionally convergent or semi convergent

If Σu_n is convergent and Σu_n is divergent.

Example:-

The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is conditional convergence.

The series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is divergent.

Theorem:

Any absolutely convergence series is convergence.

Proof:-

Let Σu_n be the gn series then by the hypothesis Σu_n is convergent

Now, $u_n + |u_n| = 2u_n$

If u_n is +ve

If u_n is -ve.

$u_n + |u_n| = 0$

Every term of the series $\Sigma(u_n + |u_n|)$ is +ve and it is $\leq$ corresponding term of the convergent series.

Hence $\Sigma u_n + |u_n|$ is convergent

We know that,

$\Sigma |u_n|$ is convergence

The series formal by try the different of the corresponding terms of the both the series is convergence

Note-01

When we say the Σu_n is an absalutly convergent we assert the convergent of another series Σu_n and not that of Σu_n alone

Note-02

If the terms of an absolutely convergent series are re-arranged the series remain convergent and sum is an alternal

Note-03

In a conditionally convergent of series there arrangement of terms may alter

Note-04

If the two series are absolutely convergent they can be multiply and resulting series is also an absolutely

Theorem 1:-

If $u_1-u_2+u_3-\cdots$ is a series of terms alternatingvely +ve and -ve if $u_n > u_{n+1}$ $\forall n$ and if $\lim_{n\to\infty} u_n = 0$. Then the series is convergent

Proof:-

divergent → change the inequality

Let sum note the sum to 2^n 2n terms of the series.

$$s_{2n} = (u_1-u_2)+(u_3-u_4)+\cdots+(u_{2n-1}+u_{2n})$$

Each bracket is +ve s_{2n} increases as n increases.

That is $s_2 < s_4 < s_6 < \cdots$ without alter the gn order of the terms the sum s_{2n} may be return in the term.

$$s_{2n} = u_1 - (u_2-u_3) - (u_4-u_5) - \cdots - (u_{n-2}-u_n) - \cdots 2n$$

Each product is +ve then $s_{2n} < u_1$

$\lim_{n\to\infty} s_{2n}$ exist and equals l, where $l < u_1$.

But $s_{2n+1} = s_{2n} + u_{2n+1}$ and $\lim_{n\to\infty} u_{2n+1} = 0$.

$\lim_{n\to\infty} s_{2n+1} = l$.

The above series is convergence.

∴ hence the proof.

An alternative series $u_1 - u_2 + u_3 - u_4 + \dots$ is convergent

i) If each term is numerically less than its presenting term and $\lim_{n\to\infty} u_n = 0$.

ii) If $\lim_{n\to\infty} u_n \neq 0$ the gn series is not convergent (or) oscillating.

Discuss the convergency of the following series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

Soln:-

The gn series to the terms are alternative +ve and -ve each term is numerically less than its presending.

i.e) $1 > \frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \dots$

Let u_n be the n^{th} term of the series.

$u_n = \frac{1}{n}$

taking lim on both sides,

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n}$

$= \frac{1}{\infty}$

$= 0$

$\therefore$ The gn series is convergence series the alternative series is convergence.

Discuss the convergency of the following series $2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \dots$

Soln:-

The terms are alternatively +ve and

the convergent if the series

$a+b+a^2+b^2+a^3+b^3+\dots$

Soln:

$$u_n = a+b+a^2+b^2+a^3+b^3+\dots$$

$$u_1 = a,\ u_3 = a^2,\ u_5 = a^3 \dots u_{2m-1} = a^m$$

$$u_2 = b,\ u_4 = b^2,\ u_6 = b^3 \dots u_{2m} = b^m$$

Let, $n = 2m-1$

$$2m = n+1$$

$$m = \frac{n+1}{2}$$

$$u_{2m-1} = a^m$$

$$u_{2\left(\frac{n+1}{2}\right)-1} = a^{\frac{n+1}{2}}$$

$$u_n = a^{\frac{n+1}{2}} \longrightarrow (1)$$

$$u_{2m} = b^m$$

Let, $n = 2m$, $m = n/2$

$$u_{2(n/2)} = b^{n/2}$$

$$u_n = b^{n/2} \longrightarrow (2)$$

From (1) and (2)

If n is odd,

$$u_n = a^{\frac{n+1}{2}}$$

$$(u_n)^{1/n} = \left(a^{\frac{n+1}{2}}\right)^{1/n}$$

$$(u_n)^{1/n} = a^{\left(\frac{1+1/n}{2}\right)}$$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} a^{\left(\frac{1+1/n}{2}\right)}$$

$$= a^{1/2}$$

n is even

$u_n = b^{n/2}$

$(u_n)^{1/n} = (b^{n/2})^{1/n}$

$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} b^{1/2}$

$= b^{1/2}$

In Cauchy's root test $a^{1/2} < 1$ and $b^{1/2} < 1$ is convergent and $a^{1/2} > 1$ and $b^{1/2} > 1$ is divergent. $a^{1/2} = 1$

i.e) $a = 1$. Sub in Σu_n.

$\Sigma u_n = 1+b+1+b^2+1+b^3+\cdots$

$= [1+1+1+1+\cdots] + [b+b^2+b^3+\cdots]$

$\therefore$ The series is divergent.

$b^{1/2} = 1$

ie) $b = 1$

$\Sigma u_n = a+1+a^2+1+a^3+1+\cdots$

$= [a+a^2+a^3+\cdots] + [1+1+1+\cdots]$

$\therefore$ The series is divergent.

Test the series $\Sigma \frac{1}{(\log n)^n}$.

Soln:-

$u_n = \frac{1}{(\log n)^n}$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{(\log n)^n}$

$= \frac{1}{(\log \infty)^\infty}$

$= 1/0$

$= \infty$

$\therefore$ The series is divergent