Unit - III

Some theorm on limits:

Theorm: 1 (Cauchey's first limit theorm).

If $a_n \to l$ then $\left(\frac{a_1+a_2+\dots+a_n}{n}\right) \to l$.

proof:

Case(i).

Let $l=0$,

Let $b_n = \frac{a_1+a_2+\dots\dots+a_n}{n}$

Let $\varepsilon>0$ be given. Since, $(a_n) \to 0$ There exists $m \in N$ such that,

$|a_n| < \frac{1}{2}\varepsilon \quad \forall n>m \longrightarrow (1)$

Now, Let $n \geq m$.

Then

$$|b_n| = \left|\frac{a_1+a_2+\dots\dots+a_m+a_{m+1}+\dots+a_n}{n}\right|$$

$$\leq \left|\frac{a_1+a_2+\dots+a_m}{n}\right| + \left|\frac{a_{m+1}+a_{m+2}+\dots+a_n}{n}\right|$$

$$= \frac{k}{n} + \frac{|a_{m+1}|+\dots+|a_n|}{n} \text{ where } n \geq m$$

$k = |a_1| + |a_2| + \dots + |a_m|$

$$< \frac{k}{n} + \left(\frac{n-m}{n}\right)\frac{\varepsilon}{2} \quad (\text{by } (1))$$

$$< \frac{k}{n} + \frac{\varepsilon}{2} \quad \left(\text{since } \frac{n-m}{n} < 1\right) \longrightarrow (2)$$

if given $\epsilon > 0$, there exists infinite number of terms of the sequence in $(a-\epsilon, a+\epsilon)$.

Theorem: 11

Every bounded sequence has atleast one limit point.

proof:

Let (a_n) be a bounded sequence. Then there exists a converge subsequence (a_{n_k}) of (a_n) converging to l (say) (by theorem)

Hence l is a limit point of (a_n).

Note: In general every sequence (a_n) has atleast one limit point (finite or infinite).

Cauchy sequence:

A sequence (a_n) is said to be a Cauchy sequence if given $\epsilon > 0$. there exists $n_0 \in N$ such that $|a_n - a_m| < \epsilon$ for all $n, m \geq n_0$.

Note: In the above definition the condition $|a_n - a_m| < \epsilon$ for all $n, m \geq n_0$ can a monotonic increasing subsequence (a_{n_k}) of (a_n).

Theorem : 9

Every bounded sequence has a convergent subsequence.

proof:

Let (a_n) be a bounded sequence.

Let (a_{n_k}) be a monotonic sub sequence of (a_n).

Since (a_n) is bounded (a_{n_k}) is also bounded therefore (a_{n_k}) is bounded monotonic sequence and hence converges.

$\therefore (a_{n_k})$ is a convergent subsequence of (a_n).

LIMIT POINTS:

Definition:

Let (a_n) be a sequence of real numbers a is called a limit point or a cluster point of the sequence (a_n)

Theorem : 8

Every sequence (a_n) has a monotonic sub sequence.

proof :

Case(i) :

(a_n) has infinite number of peak points.

Let the peak points be

$n_1 < n_2 < \dots < n_k < \dots$

Then $a_{n_1} > a_{n_2} > \dots > a_{n_k} > \dots$

$\therefore (a_{n_k})$ is a monotonic decreasing sub sequence of (a_n).

Case (ii) :

(a_n) has only a finite number of peak point, or no peak point. choose a natural n_1 such that there is no peak point. greater than or equal to n_1.

Since n_1 is not a peak point of (a_n).

there exists $n_2 > n_1$ such that $a_{n_2} \geq a_{n_1}$.

Again since n_2 is not a peak point,

there exists $n_3 > n_2$ such that $a_{n_3} \geq a_{n_2}$.

Repeating this process we get a

Theorem : 8

Every sequence (a_n) has a monotonic sub sequence

proof :

Case (i) :

(a_n) has infinite number of peak points.

Let the peak points be

$n_1 < n_2 < \dots < n_k < \dots$

Then $a_{n_1} > a_{n_2} > \dots > a_{n_k} > \dots\dots$

$\therefore (a_{n_k})$ is a monotonic decreasing sub sequence of (a_n).

Case (ii) :

(a_n) has only a finite number of peak point. or no peak point. choose a natural n_1 such that there is no peak point. greater than or equal to n_1.

Since n_1 is not a peak point of (a_n). there exists $n_2 > n_1$ such that $a_{n_2} \geq a_{n_1}$.

Again since n_2 is not a peak point. there exists $n_3 > n_2$ such that $a_{n_3} \geq a_{n_2}$.

Repeating this process we get a

Note :

The above result is true even if we have $l = \infty$ or $-\infty$.

Definition :-

Let (a_n) be a sequence. A natural number m is called a peak point of the sequence (a_n) if $a_n < a_m$ $\forall$ $n > m$.

Examples :

=> 1. For the sequence $(1/n)$, every natural number is a peak point and hence the sequence has infinite number of peak points. In general for a strictly monotonic decreasing sequence every natural number is a peak point.

=> 2. Consider the sequence $1, 1/2, 1/3, -1, -1, \ldots$ Here $1, 2, 3$ are the peak points of the sequence.

=> 3. The sequence $1, 2, 3, \ldots$ has no peak point. In general a monotonic increasing sequence has no peak point.

Then, $\lim_{n \to \infty} \left(\frac{1}{a_{n+1}}\right) / \left(\frac{1}{a_n}\right) = 0$

$\therefore$ By case (i), $\left(\frac{1}{a_n}\right)^{1/n} \longrightarrow 0$.

$\therefore \left(a_n^{1/n}\right) \longrightarrow \infty$.

Theorem : 4

Let (a_n) be any sequence and $\lim_{n \to \infty} \left|\frac{a_n}{a_{n+1}}\right| = l$. If $l > 1$ then $(a_n) \to 0$.

proof:

Let k be any real number such that ~~1<k<l~~. ~~1<k~~ $1 < k < l$.

Since $\lim_{n \to \infty} \left|\frac{a_n}{a_{n+1}}\right| = l$. there exists $m \in N$ such that,

$$l - \epsilon < \frac{a_n}{a_{n+1}} < l + \epsilon \ \forall \ n \geq m$$

choosing, $\epsilon = l - k$ we obtain $\left|\frac{a_n}{a_{n+1}}\right| > k \ \forall n \geq m$.

Now, fix $n \geq m$ then

$$\left|\frac{a_m}{a_{m+1}}\right| > k$$

$$\left|\frac{a_{m+1}}{a_{m+2}}\right| > k$$

Let $\epsilon > 0$ be any given real number. Then there exists $m \in N$ such that,

$$l - \frac{1}{2}\epsilon < \frac{a_{n+1}}{a_n} < l + \frac{1}{2}\epsilon \quad \forall \ n \geq m$$

Now choose $n \geq m$.

Then,

$$l - \frac{1}{2}\epsilon < \frac{a_{m+1}}{a_m} < l + \frac{1}{2\epsilon}$$

$$l - \frac{1}{2}\epsilon < \frac{a_{m+2}}{a_{m+1}} < l + \frac{1}{2}\epsilon$$

....

....

$$l - \frac{1}{2}\epsilon < \frac{a_n}{a_{n-1}} < l + \frac{1}{2}\epsilon$$

Multiplying these inequalities, we obtain

$$\left(l - \frac{1}{2}\epsilon\right)^{n-m} < \frac{a_n}{a_m} < \left(l + \frac{1}{2}\epsilon\right)^{n-m}$$

$$\left(l - \frac{1}{2}\epsilon\right)^{n}\left(l - \frac{1}{2}\epsilon\right)^{-m} < \frac{a_n}{a_m} < \left(l + \frac{1}{2}\epsilon\right)^{n}\left(l + \frac{1}{2}\epsilon\right)^{-m}$$

$$\therefore a_m \frac{\left(l - \frac{1}{2}\epsilon\right)^{n}}{\left(l - \frac{1}{2}\epsilon\right)^{m}} < a_n < a_m \frac{\left(l + \frac{1}{2}\epsilon\right)^{n}}{\left(l + \frac{1}{2}\epsilon\right)^{m}}$$

$$\therefore K_1\left(l - \frac{1}{2}\epsilon\right)^{n} < a_n < K_2\left(l + \frac{1}{2}\epsilon\right)^{n}$$

$\therefore k_1^{1/n}\left(l - \tfrac{1}{2}\varepsilon\right) < a_n^{1/n} < k_2^{1/n}\left(l + \tfrac{1}{2}\varepsilon\right)$

Now,

$$\left(k_1^{1/n}\left(l - \tfrac{1}{2}\varepsilon\right)\right) \to l - \tfrac{1}{2}\varepsilon \quad \left(\text{since } (k_1^{1/n}) \to 1.\right)$$

$\therefore$ There exists $n_1 \in N$ such that

$$\left(l - \tfrac{1}{2}\varepsilon\right) - \tfrac{1}{2}\varepsilon < k_1^{\frac{1}{n}}\left(l - \tfrac{1}{2}\varepsilon\right) < \left(l - \tfrac{1}{2}\varepsilon\right) + \tfrac{1}{2}\varepsilon$$

for all $n \geq n_1$ $\longrightarrow (2)$

III^{ly}, there exists $n_2 \in N$ such that,

$$\left(l + \tfrac{1}{2}\varepsilon\right) - \tfrac{1}{2}\varepsilon < k_2^{1/n}\left(l + \tfrac{1}{2}\varepsilon\right) < \left(l + \tfrac{1}{2}\varepsilon\right) + \tfrac{1}{2}\varepsilon \quad \forall n \geq n_2 \longrightarrow (3)$$

Let $n_0 = \max\{m, n_1, n_2\}$

Then

$$l - \varepsilon < k_1^{1/n}\left(l - \tfrac{1}{2}\varepsilon\right) < a_n^{1/n} < k_2^{1/n}\left(l + \tfrac{1}{2}\varepsilon\right) < l + \varepsilon \text{ for all } n \geq n_0$$

(by 1, 2, 3)

$\therefore l - \varepsilon < a_n^{1/n} < l + \varepsilon \ \forall n \geq n_0.$

Hence, $(a_n^{1/n}) \to l.$

<u>Case (ii)</u>

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \infty.$$

$$b_n = \frac{a_1 + a_2 + \cdots + a_n}{n} = \begin{cases} 0 & \text{if } n \text{ is even} \\ -1/n & \text{if } n \text{ is odd} \end{cases}$$

clearly $(b_n) \to 0$ and (a_n) is not convergent.

Theorm: 2 (Cesaro's theorm).

if $(a_n) \to a$ and $(b_n) \to b$ Then

$$\left(\frac{a_1 b_n + a_2 b_{n-1} + \cdots a_n b_1}{n}\right) \to ab.$$

proof:

Let, $C_n = \dfrac{a_1 b_n + \cdots\cdots + a_n b_1}{n}$

Now, put $a_n = a + r_n$, so that

$(r_n) \to 0$.

Then $C_n = \dfrac{(a + r_1) b_n + \cdots\cdots + (a + r_n) b_1}{n}$

$$= \frac{a(b_1 + \cdots\cdots + b_n)}{n} + \frac{r_1 b_n + \cdots + r_n b_1}{n}$$

Now, by Cauchy's first limit theorm.

$$\left(\frac{b_1 + b_2 + \cdots\cdots + b_n}{n}\right) \to b.$$

$$\therefore \left(\frac{a(b_1 + b_2 + \cdots\cdots + b_n)}{n}\right) \to ab.$$

Hence it is enough if we prove, that

$$\left(\frac{r_1 b_n + \cdots\cdots + r_n b_1}{n}\right) \to 0.$$

Now, since $(b_n) \to b$, (b_n) is a bounded sequence.

$\therefore$ There exists a real number $k > 0$ such that $|b_n| \leq k \ \forall n$.

$$\therefore \left| \frac{r_1 b_n + \cdots + r_n b_1}{n} \right| \leq k \left| \frac{r_1 + \cdots + r_n}{n} \right|.$$

Since $(r_n) \to 0$, $\left(\frac{r_1 + \cdots + r_n}{n} \right) \to 0$ (by theorm (1))

$$\left(\frac{r_1 b_n + \cdots + r_n b_1}{n} \right) \to 0.$$

Hence the theorem.

Theorem : 3

Let (a_n) be a sequence of positive terms. Then $\lim\limits_{n \to \infty} a_n^{1/n} = \lim\limits_{n \to \infty} \frac{a_{n+1}}{a_n}$ provided the limit on the right hand side exists, whether finite (or) infinite.

proof:

Case (i)

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = l \text{ finite.}$$

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can be written in the following equivalent form namely $|a_{n+p} - a_n| < \varepsilon$ for all $n \geq n_0$ and for all positive integers p.

Examples:

1. The sequence $(1/n)$ is a Cauchy sequence.

proof:

Let $(a_n) = (1/n)$.

Let, $\varepsilon > 0$ be given.

Now, $|a_n - a_m| = \left|\frac{1}{n} - \frac{1}{m}\right|$

$\therefore$ If we choose n_0 to be any positive integer greater than $\frac{1}{\varepsilon}$ we get,

$|a_n - a_m| < \varepsilon \;\; \forall \; n, m \geq 0$.

$\therefore (1/n)$ is a Cauchy sequence.

2. The sequence $((-1)^n)$ is not a Cauchy sequence.

proof.

Let $(a_n) = ((-1)^n)$

$\therefore |a_n - a_{n+1}| = 2$

Note:

The above theorem is true even if $l = \infty$.

Theorem: 5

If the sequence (a_n) and (b_n) converge to 0 and (b_n) is strictly monotonic decreasing $\lim_{n \to \infty} \left(\frac{a_n}{b_n} \right) = \lim_{n \to \infty} \left(\frac{a_n - a_{n+1}}{b_n - b_{n+1}} \right)$ provided the limit on the right hand side exists whether finite or infinite.

proof:

Case (i)

Let $\lim_{n \to \infty} \left(\frac{a_n - a_{n+1}}{b_n - b_{n+1}} \right) = l$. finite

Let $\epsilon > 0$ be given.

Then there exists $m \in N$, such that,

$$l - \epsilon < \frac{a_n - a_{n+1}}{b_n - b_{n+1}} < l + \epsilon \ \forall \ n \geq m.$$

Since $b_n - b_{n+1} > 0$, we get

$$(b_n - b_{n+1})(l - \epsilon) < a_n - a_{n+1} < (b_n - b_{n+1})(l + \epsilon) \ \forall \ n \geq m$$

Let $n > p \geq m$.

Then,

$(b_p - b_{p+1})(l-\varepsilon) < a_p - a_{p+1} < (b_p - b_{p+1})(l+\varepsilon)$

$(b_{p+1} - b_{p+2})(l-\varepsilon) < a_{p+1} - a_{p+2} < (b_{p+1} - b_{p+2})(l+\varepsilon)$

$\ldots\ldots\ldots\ldots\ldots\ldots$

$\ldots\ldots\ldots\ldots\ldots\ldots$

$(b_{n-1} - b_n)(l-\varepsilon) < a_{n-1} - a_n < (b_{n-1} - b_n)(l+\varepsilon)$

Adding the above inequalities.

we get.

$(b_p - b_n)(l-\varepsilon) < a_p - a_n < (b_p - b_n)(l+\varepsilon)$.

Taking limit as $n \to \infty$, we get

$b_p(l-\varepsilon) < a_p < b_p(l+\varepsilon)$ $[\because (a_n), (b_n) \to 0)$.

$\therefore l - \varepsilon < \dfrac{a_p}{b_p} < l+\varepsilon$ $(\because (b_p > 0))$

$\therefore \left| \dfrac{a_p}{b_p} - l \right| < \varepsilon \ \forall \ p \geq m$.

$\therefore \lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = l$

Case (ii)

$$\lim_{n\to\infty} \left(\frac{a_n - a_{n+1}}{b_n - b_{n+1}} \right) = \infty.$$

Let $k > 0$ be any real number. Then there exists $m \in N$ such that $\dfrac{a_n - a_{n+1}}{b_n - b_{n+1}} > k$ $\forall \ n \geq m$.

$\therefore a_n - a_{n+1} > (b_n - b_{n+1})k \;\; \forall\; n \geq m.$

Let $n > p \geq m$.

writing the inequalities for $n = p, p+1, \ldots, n$ and adding we get

$$a_p - a_n > k(b_p - b_n)$$

Taking limit as $n \to \infty$, we get

$$a_p \geq k\, b_p$$

$$\therefore \frac{a_p}{b_p} \geq k \quad \forall\; p \geq m$$

$$\therefore \left(\frac{a_n}{b_n}\right) \text{ diverges to } \infty.$$

Hence the proof.

1. Show that $\lim\limits_{n \to \infty} \frac{1}{n}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) = 0.$

Soln:

Let $a_n = \frac{1}{n}$

$\therefore (a_n) \to 0$. Hence by Cauchy's first limit theorem, we get

$$\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \to 0.$$

$$\therefore \left(\frac{1}{n}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)\right) \to 0 \text{ (or)}$$

$$\lim \frac{1}{n}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) = 0.$$

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Now, since $\left(\frac{k}{n}\right) \to 0$, There exists $n_0 \in N$ such that

$$\frac{k}{n} < \frac{1}{2}\varepsilon \quad \forall \ n \geq n_0 \qquad \longrightarrow (3)$$

Let $\quad n_1 = \max\{m, n_0\}$

Then $|b_n| < \varepsilon$ for all $n \geq n_1$ (using 2 & 3)

$$\therefore (b_n) \to 0.$$

Case (ii)

Let $\quad l \neq 0$

since $(a_n) \to l$, $(a_n - l) \to 0$

$$\therefore \left(\frac{(a_1 - l) + (a_2 - l) + \dots + (a_n - l)}{n}\right) \to 0$$

(by case (i))

$$\therefore \left(\frac{a_1 + a_2 + \dots + a_n - nl}{n}\right) \to 0$$

$$\therefore \left(\frac{a_1 + a_2 + \dots + a_n}{n} - l\right) \to 0$$

$$\therefore \left(\frac{a_1 + a_2 + \dots + a_n}{n}\right) \to l.$$

Note :

The converse of the above theorem is not true. For example, consider the sequence $(a_n) = ((-1)^n)$.

Example : 3

Any sequence (a_n) is a sub sequence of itself.

Example : 4

Consider the sequence (a_n) given by $1, 0, 1, 0, \ldots$ Now, (b_n) given by $1, 1, 1, \ldots$ is a sub sequence of (a_n). Here (a_n) is not convergent where as the sub sequence (b_n) converges to 1. Thus a sub sequence of non-convergent sequence can be a convergent sequence.

Note :

A sub sequence of a given sub sequence (a_{n_k}) of a sequence (a_n) is again a sub sequence of (a_n).

Theorem : 6

If a sequence (a_n) converges to l. then every sub sequence (a_{n_k}) of (a_n) also converges to l.

proof :

Let $\epsilon > 0$ be given

$$\left|\frac{a_{n-1}}{a_n}\right| > k \quad \forall\ n \geq m$$

Multiplying the above inequalities, we get

$$\left|\frac{a_m}{a_n}\right| > k^{n-m}$$

$$\left|\frac{a_m}{a_n}\right| > \frac{k^n}{k^m}$$

$$k^m |a_m| > |a_n| k^n$$

$$\frac{k^m}{k^n} > \left|\frac{a_n}{a_m}\right|$$

$$k^m \left(\frac{1}{k}\right)^n > \left|\frac{a_n}{a_m}\right|$$

$$\left|\frac{a_n}{a_m}\right| < k^m \left(\frac{1}{k}\right)^n$$

$$|a_n| < |a_m| k^m \left(\frac{1}{k}\right)^n$$

$|a_n| < A.r^n$ where $A = k^m |a_m|$ is a constant and $r = 1/k$

Now, $k > 1 \Rightarrow 0 < r < 1$

$\therefore (r^n) \to 0$

$\therefore (a_n) \to 0.$

Since $(a_n) \to l$ there exists $m \in N$ such that

$|a_n - l| < \varepsilon$ for all $n \geq m$ $\longrightarrow (1)$

Now, choose $n_{k_0} \geq m$;

Then $k \geq k_0 \Rightarrow n_k \geq n_{k0}$

(since (n_k) is a monotonic increasing)

$\Rightarrow n_k \geq m$.

$\Rightarrow |a_{n_k} - l| < \varepsilon$ (by (1))

Thus $|a_{n_k} - l| < \varepsilon$ $\forall$ $k \geq k_0$.

$\therefore (a_{n_k}) \to l$.

Note : 1

If a subsequence of a sequence converges, then the original converges, then the original sequence need not converge (refer eg. 4)

Note : 2

If a sequence (a_n) has two subsequences converging to two different limits. then (a_n) does not converge

For example, consider the sequence (a_n)

given by

$$a_n = \begin{cases} 1/n & \text{if } n \text{ is even} \\ 1+1/n & \text{if } n \text{ is odd} \end{cases}$$

Here the subsequence $(a_{2n}) \to 0$ and the subsequence $(a_{2n-1}) \to 1$. Hence the sub sequence (a_n) does not converge.

Theorem: 7

If the subsequences (a_{2n-1}) and (a_{2n}) of a sequence (a_n) converge to the same limit l then (a_n) also converges to l.

Proof:

Let $\varepsilon > 0$ be given.

Since, $(a_{2n-1}) \to l$ there exists $n_1 \in N$. such that

$$|a_{2n-1} - l| < \varepsilon \quad \forall \; 2n-1 \geq n_1.$$

Similarly, there exists $n_2 \in N$. such that

$$|a_{2n} - l| < \varepsilon \quad \forall \; 2n \geq n_2.$$

Let, $m = \max\{n_1, n_2\}$.

clearly, $|a_n - l| < \varepsilon \;\; \forall \; n \geq m$.

$\therefore (a_n) \to l$.

SUBSEQUENCES:

Definition:

Let (a_n) be a sequence. Let (n_k) be a strictly increasing sequence of natural numbers. Then (a_{n_k}) is called a subsequence of (a_n).

Example : 1

(a_{2n}) is a subsequence of any sequence (a_n). Note that in this example the interval between any two terms of the subsequence is the same (i.e.,) $n_1 = 2$, $n_2 = 4$, $n_3 = 6, \ldots\ldots n_k = 2k$.

Example : 2

(a_{n^2}) is a subsequence of any sequence (a_n). Hence $a_{n_1} = a_1$, $a_{n_2} = a_4$, $a_{n_3} = a_9 \ldots\ldots$ Here the internal between two successive terms of the subsequence goes on increasing as k becomes goes on increasing as k becomes large. Thus the internal between various terms of a subsequence need not be regular.

$$\therefore \frac{a_n}{a_{n+1}} = \frac{x^n}{n} \cdot \frac{(n+1)!}{x^{n+1}}$$

$$= \frac{n+1}{x}$$

$$\therefore \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \infty$$

$\therefore (a_n) \to 0$ [by theorem (4.1)]

5. Show that $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

soln:-

Let, $a_n = \frac{n!}{n^n}$

$$a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$$

$$\therefore \left|\frac{a_n}{a_{n+1}}\right| = \frac{n!}{n^n} \cdot \frac{(n+1)^{n+1}}{(n+1)!}$$

$$= \frac{n!}{n^n} \frac{(n+1)^n (n+1)}{(n+1)!}$$

$$= \frac{(n+1)^n}{n^n}$$

$$= \left(1+1/n\right)^n$$

$$\therefore \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left(1+1/n\right)^n$$

$$= e > 1.$$

$\therefore (a_n) \to 0$.

2. Show that $\lim_{n\to\infty} n^{1/n} = 1$.

Soln:

Let $a_n = n$

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{n}$$

$$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{n(1+1/n)}{n}$$

$$= \lim_{n\to\infty} (1+1/n)$$

$$= 1 + 1/\infty$$

$$= 1$$

$\therefore$ By Cauchy's second limit theorem, we get

$$\lim_{n\to\infty} n^{1/n} = 1.$$

4. prove that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$.

soln:

Let, $a_n = \frac{x^n}{n!}$

$$a_{n+1} = \frac{x^{n+1}}{(n+1)!}$$

$\therefore$ If $\epsilon < 2$, we cannot find n_0 such that $|a_n - a_{n+1}| < \epsilon$ for all $n \geq n_0$.

$\therefore ((-1)^n)$ is not a Cauchy sequence.

Theorem : 16

(Cauchy's general principle of convergence).

A sequence (a_n) is R is convergent iff it is a cauchy sequence.

proof :

In theorem 13 we have proved that any convergent sequence is a Cauchy sequence.

conversely, let (a_n) be a Cauchy sequence in R.

$\therefore (a_n)$ is a bounded sequence

(by theorem 4)

$\therefore$ There exists a sub sequence.

(a_{n_k}) of (a_n) such that $(a_{n_k}) \to l$.

$\therefore (a_n) \to l$ (by theorem 15).

Note:

There are Cauchy sequences in Q which are not convergent in Q. for example, the sequence $1, 1.4, 1.41, 1.414, \ldots$ whose terms are successive decimal expressions of $\sqrt{2}$ is a Cauchy sequence in Q which is not convergent in Q.