Unit - 2

The Algebra of Limits.

In This section we prove a few simple theorems for sequences which are very useful in calculating limits of sequences.

Theorem 3.8

If $(a_n) \to a$ and $(b_n) \to b$ then $(a_n+b_n) \to a+b$.

proof:

Let $\varepsilon > 0$ be given.

Now $|a_n+b_n-a-b| = |a_n-a+b_n-b|$

$\le |a_n-a| + |b_n-b| \quad \to (1)$

Since $(a_n) \to a$, there exists a natural number n_1 such that

$|a_n-a| < \frac{1}{2}\varepsilon$ for all $n \ge n_1$. $\longrightarrow (2)$

Since $(b_n) \to b$, there exists a natural number n_2 such that

$|b_n-b| < \frac{1}{2}\varepsilon$ for all $n \ge n_2$ $\longrightarrow (3)$

Let $m = \max\{n_1, n_2\}$.

Then $|a_n+b_n-a-b| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon =$

ε for all $n \ge m$

(by 1, 2 & 3).

$\therefore (a_n+b_n) \to a+b$.

Note : $\therefore$ Similary we can prove that $(a_n-b_n) \to a-b$.

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19

Theorem 3.9

If $(a_n) \to a$ and $k \in R$ then $(ka_n) \to ka$.

Proof:

If $k = 0$. (ka_n) is the constant sequence 0, 0, 0, ... and hence the result is trivial.

Now, let $k \neq 0$.

Then $|ka_n - ka| = |k| |a_n - a|$ $\longrightarrow (1)$

Let $\varepsilon > 0$ be given.

Since $(a_n) \to a$, there exists $m \in N$ such that

$|a_n - a| < \frac{\varepsilon}{|k|}$

for all $n \geq m$. $\longrightarrow (2)$.

$\therefore |ka_n - ka| < \varepsilon$ for all $n \geq m$ by (1 and 2).

$\therefore (ka_n) \to ka$

Theorem 3.10.

If $(a_n) \longrightarrow a$ and $(b_n) \to b$ then $(a_n b_n) \to ab$

Proof:

Let $\varepsilon > 0$ be given

Now, $|a_n b_n - ab| = |a_n b_n - a_n b + a_n b - ab|$

$\leq |a_n b_n - a_n b| + |a_n b - ab|$

$= |a_n| |b_n - b| + |b| |a_n - a|$ $\longrightarrow (1)$.

Also since $(a_n) \to a$; (a_n) is a bounded sequence.

(by theorem 3.2)

$\therefore$ There exists a real number $k > 0$ such that

$|a_n| \leq k$ for all n. $\longrightarrow (2)$.

Using (1) and (2) we get

$|a_n b_n - ab| \leq k |b_n - b| + |b| |a_n - a|$ $\longrightarrow (3)$

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Now since $(a_n) \to a$ there exists a natural number n_1 such that

$$|a_n - a| < \frac{\epsilon}{2|b|} \text{ for all } n \geq n_1 \quad \to (4)$$

Since $(b_n) \to b$, there exists a natural number n_2 such that

$$|b_n - b| < \frac{\epsilon}{2K} \text{ for all } n \geq n_2 \quad \to (5)$$

Let $m = \max\{n_1, n_2\}$. Then

$|a_n b_n - ab| < K\left(\frac{\epsilon}{2K}\right) + |b|\left(\frac{\epsilon}{2|b|}\right) = \epsilon$ for all $n \geq m$ (by 3, 4 and 5).

Hence $(a_n b_n) \to ab$.

Theorem 3.11

If $(a_n) \to a$ and $a_n \neq 0$ for all n and $a \neq 0$. Then $\left(\frac{1}{a_n}\right) \to \frac{1}{a}$.

proof:

Let $\epsilon > 0$ be given

$$\text{We have } \left|\frac{1}{a_n} - \frac{1}{a}\right| = \left|\frac{a_n - a}{a_n a}\right| = \frac{1}{|a_n||a|}|a_n - a| \quad \to (1)$$

Now, $a \neq 0$. Hence $|a| > 0$.

Since $(a_n) \to a$ there exists $n_1 \in N$ such that $|a_n - a| < \frac{1}{2}|a|$ for all $n \geq n_1$.

Hence $|a_n| > \frac{1}{2}|a|$ for all $n \geq n_1$. $\quad \to (2)$

Using (1) and (2) we get

$$\left|\frac{1}{a_n} - \frac{1}{a}\right| < \frac{2}{|a|^2}|a_n - a| \text{ for all } n \geq n_1 \quad \to (3)$$

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Now since $(a_n) \to a$ there exists $n_2 \in N$ such that

$|a_n - a| < \frac{1}{2} \varepsilon |a|^2$ for all $n \geq n_2$ ——→ (4)

Let $m = \max\{n_1, n_2\}$

$\therefore \left| \frac{1}{a_n} - \frac{1}{a} \right| < \frac{2 |a|^2 \varepsilon}{|a|^2 2} = \varepsilon$ for all $n \geq m$ (by 3 and 4)

$\therefore \left(\frac{1}{a_n} \right) \to \frac{1}{a}$

Corollary. Let $(a_n) \to a$ and $(b_n) \to b$ where $b_n \neq 0$ for all n and $b \neq 0$. Then $\left(\frac{a_n}{b_n} \right) \to \frac{a}{b}$

Proof: $\left(\frac{1}{b_n} \right) \to \frac{1}{b}$ (by theorem 3.11)

$\therefore \left(\frac{a_n}{b_n} \right) \to \frac{a}{b}$ (by theorem 3.10)

Theorem 3.12: If $(a_n) \to a$ then $(|a_n|) \to |a|$.

proof: Let $\varepsilon > 0$ be given

Now. $\big| |a_n| - |a| \big| \leq |a_n - a|$ ——→ (1)

Since $(a_n) \to a$, there exists $m \in N$ such that $|a_n - a| < \varepsilon$ for all $n \geq m$

Hence from (1) we get $\big| |a_n| - |a| \big| < \varepsilon$ for all $n \geq m$.

Hence $(|a_n|) \to |a|$.

Theorem 3.13

If $(a_n) \to a$ and $a_n \geq 0$ for all n then $a \geq 0$.

proof: suppose $a < 0$. Then $-a > 0$.

choose ε such that $0 < \varepsilon < -a$ so that $a + \varepsilon < 0$.

Now, since $(a_n) \to a$, there exists $m \in N$ such that $|a_n - a| < \varepsilon$ for all $n \geq m$.

$a - \varepsilon < a_n < a + \varepsilon$ for all $n \geq m$

Now, since $a + \varepsilon < 0$, we have $a_n < 0$ for all $n \geq m$ which is a contradiction since $a_n \geq 0$. Hence $a \geq 0$.

Theorem 3.14.

If $(a_n) \to a$, $(b_n) \to b$ and $a_n \leq b_n$ for all n, then $a \leq b$.

Proof:

Since $a_n \leq b_n$, we have $b_n - a_n \geq 0$ for all n.

Also $(b_n - a_n) \to b - a$ (by theorem 3.8)

$\therefore b - a \geq 0$ (by theorem 3.13

$\therefore a \leq b$.

Theorem 3.15.

If $(a_n) \to l$, $(b_n) \to l$ and $a_n \leq c_n \leq b_n$ for all n, then $(c_n) \to l$.

Proof.

Let $\varepsilon > 0$ be given.

Since $(a_n) \to l$, there exists $n \in N$ such that $l - \varepsilon < b_n < l + \varepsilon$ for all $n \geq n_2$

Let $m = \max\{n_1, n_2\}$.

$\therefore l - \varepsilon < a_n \leq c_n \leq b_n < l + \varepsilon$ for all $n \geq m$.

$\therefore l - \varepsilon < c_n < l + \varepsilon$ for all $n \geq m$.

$\therefore |c_n - l| < \varepsilon$ for all $n \geq m$

$\therefore (c_n) \to l$.

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Theorem 3.16

If $(a_n) \to a$ and $a_n \geq 0$ for all n and $a \neq 0$, then $(\sqrt{a_n}) \to \sqrt{a}$.

Proof:

Since $a_n \geq 0$ for all n, $a \geq 0$. (by Theorem 3.13)

Now, $|\sqrt{a_n} - \sqrt{a}| = \left| \dfrac{a_n - a}{\sqrt{a_n} + \sqrt{a}} \right|$

Since $(a_n) \to a \neq 0$, as in theorem 3.11 we obtain

$a_n > \frac{1}{2} a$ for all $n \geq n_1$.

$\therefore \sqrt{a_n} > \sqrt{(\frac{1}{2} a)}$ for all $n \geq n_1$.

$\therefore |\sqrt{a_n} - \sqrt{a}| < \dfrac{\sqrt{2}}{(\sqrt{2}+1)\sqrt{a}} |a_n - a|$ for all $n \geq n_1$. $\to$ (1)

Now let $\varepsilon > 0$ be given.

Since $(a_n) \to a$, there exists $n_2 \in N$ such that

$|a_n - a| < \varepsilon \dfrac{\sqrt{a}(\sqrt{2}+1)}{\sqrt{2}}$ for all $n \geq n_2$ $\to$ (2)

Let $m = \max\{n_1, n_2\}$

Then $|\sqrt{a_n} - \sqrt{a}| < \varepsilon$ for all $n \geq m$ (by 1 & 2)

$\therefore (\sqrt{a_n}) \to \sqrt{a}$.

Theorem 3.17

If $(a_n) \to \infty$ and $(b_n) \to \infty$ then $(a_n + b_n) \to \infty$.

Proof:

Let $k > 0$ be any given real number.

Since $(a_n) \to \infty$, there exists $n_1 \in N$ such that

$a_n > \frac{1}{2} k$ for all $n \geq n_1$.

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2ⁿᵈ similary there exists $n_2 \in N$ such that ~~$b_n > \sqrt{k}$ for all $n \geq n_2$~~ $b_n > \frac{1}{2}k$ for all $n \geq n_2$.

let $m = \max\{n_1, n_2\}$.

~~Then $a_n b_n > k$ for all $n \geq m$.~~

Then $a_n + b_n > k$ for all $n \geq m$.

$\therefore (a_n + b_n) \to \infty$.

Proof: let $k > 0$ be any given real number since $(a_n) \to \infty$. there exists $n_1 \in N$ such that $a_n > \sqrt{k}$ for all $n \geq n_1$.

similary there exists $n_2 \in N$ such that $b_n > \sqrt{k}$ for all $n \geq n_2$.

let $m = \max\{n_1, n_2\}$.

Then $a_n b_n > k$ for all $n \geq m$.

$\therefore (a_n b_n) \to \infty$.

1. show that $\lim\limits_{n \to \infty} \dfrac{3n^2 + 2n + 5}{6n^2 + 4n + 7}$.

soln:

$$a_n = \lim_{n \to \infty} \frac{3n^2 + 2n + 5}{6n^2 + 4n + 7}$$

$$= \lim_{n \to \infty} \frac{\cancel{n^2}\left(3 + \frac{2}{n} + \frac{5}{n^2}\right)}{\cancel{n^2}\left(6 + \frac{4}{n} + \frac{7}{n^2}\right)}$$

$$= \lim_{n \to \infty} \frac{3 + \frac{2}{n} + \frac{5}{n^2}}{6 + \frac{4}{n} + \frac{7}{n^2}}$$

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(2)

$$a_n = \frac{3+\frac{2}{\infty}+\frac{5}{\infty}}{6+\frac{4}{\infty}+\frac{7}{\infty}}$$

$$= \frac{3}{6}$$

$$= \frac{1}{2}$$

2. show that $\lim\limits_{n\to\infty} \frac{1^2+2^2+\cdots+n^2}{n^3}$

soln:

$$a_n = \lim_{n\to\infty} \frac{1^2+2^2+\cdots+n^2}{n^3}$$

$$\because 1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$a_n = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3}$$

$$= \lim_{n\to\infty} \frac{n^2\left(1+\frac{1}{n}\right)n\left(2+\frac{1}{n}\right)}{6n^3}$$

$$= \lim_{n\to\infty} \frac{n^3\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6n^3}$$

$$= \frac{\left(1+\frac{1}{\infty}\right)\left(2+\frac{1}{\infty}\right)}{6} \quad \left[\because \frac{1}{\infty}=0\right]$$

$$= \frac{(1)(2)}{6} = \frac{2}{6}$$

$$= \frac{1}{3}$$

3. show that $\lim\limits_{n\to\infty} \frac{n}{\sqrt{n^2+1}}$

soln:

$$a_n = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}}$$

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$$= \lim_{n\to\infty} \frac{n}{n\sqrt{1+\frac{1}{n^2}}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}}$$

$$= \frac{1}{\sqrt{1+\frac{1}{\infty}}}$$

$$= \frac{1}{\sqrt{1}}$$

$a_n = 1$.

3.7. Behaviour of MONOTONIC sequence:

Theorem 3.21

(i) A monotonic increasing sequence which is bounded above converges to its l.u.b.

(ii) A monotonic increasing sequence which is not bounded above converges to ∞.

(iii) A monotonic decreasing sequence which is bounded below converges to its g.l.b

iv). A monotonic decreasing sequence which is not bounded below diverges to $-\infty$

Proof:-

(i). Let (a_n) be a monotonic increasing sequence which is bounded above. Let k be the l.u.b of the sequence.

Then $a_n \le k$ for all n. $\longrightarrow (1)$

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2→ Now, let $\epsilon > 0$ be given

$\therefore K - \epsilon < K$ and hence $K - \epsilon$ is not an upper bound of (a_n).

Hence, there exists a_m such that $a_m > K - \epsilon$.

Now, since (a_n) is monotonic increasing, $a_n \geq a_m$ for all $n \geq m$. $\longrightarrow (2)$

Hence $a_n > K - \epsilon$ for all $n \geq m$

$\therefore K - \epsilon < a_n \leq K$ for all $n \geq m$ (by (1) and (2))

$\therefore |a_n - K| < \epsilon$ for all $n \geq m$.

$\therefore (a_n) \to K$.

(ii). Let (a_n) be a monotonic increasing sequence which is not bounded above.

Let $K > 0$ be any real number.

Since (a_n) is not bounded, there exists $m \in N$ such that $a_m > K$. Also $a_n \geq a_m$ for all $n \geq m$.

$\therefore a_n > K$ for all $n \geq m$.

$\therefore (a_n) \to \infty$.

Proof of (iii) is similar to that of (i).

Proof of (iv) is similar to that of (ii)

Problem .1

Let $a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}$. Show that $\lim_{n\to\infty} a_n$ exists lies between 2 and 3.

Soln:.

Clearly (a_n) is a monotonic increasing sequence

Also, $a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}$

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$$\leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}}$$

$$= 1 + \left(\frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} \right)$$

$$= 1 + 2\left(1 - \frac{1}{2^n}\right)$$

$$= 3 - \frac{1}{2^{n-1}} < 3.$$

$\therefore a_n < 3$

$\therefore (a_n)$ is bounded above

$\therefore \lim_{n\to\infty} a_n$ exists

Also $2 < a_n < 3$ for all n

$$\therefore 2 \leq \lim_{n\to\infty} a_n \leq 3.$$

Hence the result.

Note : The limit of the above sequence is denoted by e.

Problem 2.

show that the sequence $\left(1 + \frac{1}{n}\right)^n$ converges.

Soln::

Let $a_n = \left(1 + \frac{1}{n}\right)^n$. By binomial theorem

$$a_n = 1 + 1 + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + \dots + \frac{1}{n^n}$$

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$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) +$

$\ldots\ldots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{n-1}{n}\right)$

$< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}$

< 3 (refer problem 1).

$\therefore (a_n)$ is bounded above.

Also,

$a_{n+1} = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n+1}\right) + \frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)$

$+ \ldots\ldots + \left(\frac{1}{(n+1)!}\right)\left(1-\frac{1}{n+1}\right)\ldots\left(1-\frac{n}{n+1}\right)$

$> 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \ldots + \frac{1}{n!}$

$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{n-1}{n}\right)$

$\therefore a_{n+1} > a_n$

$\therefore (a_n)$ is monotonic increasing

$\therefore (a_n)$ is a convergent sequence.

Problem 3.

Show that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{1!}+\ldots+\frac{1}{n!}\right)$

soln:

Let $a_n = \left(1+\frac{1}{n}\right)^n$ and $b_n = 1 + \frac{1}{1!} + \ldots + \frac{1}{n!}$

Then $a_n < b_n$ for all n (refer problem 2 above)

$\therefore \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$ ——— (1)

Now, let $m > n$.

$$a_m = \left(1+\frac{1}{m}\right)^m$$

$$= 1+1+\frac{1}{2!}\left(1-\frac{1}{m}\right)+\frac{1}{3!}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)+\dots+\frac{1}{m!}\left(1-\frac{1}{m}\right)\dots$$

$$+\frac{1}{n!}\left(1-\frac{1}{m}\right)\dots\left(1-\frac{n-1}{m}\right)+\dots$$

$$\left(1-\frac{m-1}{m}\right) > 1+1+\frac{1}{2!}\left(1-\frac{1}{m}\right)+\dots+\frac{1}{n!}\left(1-\frac{1}{m}\right)\dots\left(1-\frac{n-1}{m}\right)$$

Fixing n and taking limits as $m \to \infty$ we get

$$\lim_{m\to\infty} a_m \geq 1+1+\frac{1}{2!}+\dots+\frac{1}{n!} = b_n$$

Now taking limits as $n \to \infty$ we get

$$\lim_{m\to\infty} a_m \geq \lim_{m\to\infty} b_n \longrightarrow (2)$$

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = e \quad \text{(by (1) (and (2))}$$

Problem 4.

Let $a_n = \frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{n+n}$ show that (a_n) converges.

Soln:

$$a_{n+1} - a_n = \left(\frac{1}{n+2}+\dots+\frac{1}{2n+2}\right) - \left(\frac{1}{n+1}+\dots+\frac{1}{n+n}\right)$$

$$= \frac{1}{2n+1}+\frac{1}{2n+2}-\frac{1}{n+1}$$

$$= \frac{1}{2n+1}-\frac{1}{2n+2} > 0 \text{ for all } n$$

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$\therefore a_{n+1} > a_n$ for all n.

$\therefore (a_n)$ is a monotonic increasing sequence.

Also $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n}$

$< \frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n} = 1$ for all n.

$\therefore (a_n)$ is bounded above.

$\therefore (a_n)$ converges.

problem 5.

Let $(a_n) = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$. Show that (a_n) diverges to ∞.

soln::

clearly (a_n) is a monotonic increasing sequence

Now, let $m = 2^n - 1$.

$a_m = 1 + \frac{1}{2} + \dots + \frac{1}{2^n - 1}$

$= 1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \dots + \left(\frac{1}{2^{n-1}} + \dots + \frac{1}{2^n - 1}\right)$

$> 1 + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \dots + \left(\frac{1}{2^n} + \dots + \frac{1}{2^n}\right)$

$= 1 + (n-1)\frac{1}{2}$

$= \frac{1}{2}(n+1)$

$\therefore a_m > \frac{1}{2}(n+1)$

$\therefore (a_n)$ is not bounded above.

Hence $(a_n) \to \infty$.

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