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Sequence's

Definition:-

Let $f : N \to R$ be a function and let $f(n) = a_n$. Then $a_1, a_2, a_3, \ldots a_n \ldots$ is called the sequence in R.

Determined by the function f had is denoted by (a_n).

(a_n) is called the n^{th} terms of the sequence. The range of the function f, which is a sub set of R is called the range of the sequence.

Examples:-

1. The function $f : N \to R$ given by $f(n) = n$ determines the sequences $1, 2, 3, 4, \ldots n.$

2. The function $f : N \to R$ given by $f(n) = n^2$ determines sequence $1, 4, 9, \ldots n^2$

3. The function $f : N \to R$ given by $f(n) = (-1)^n$ determines the sequence $-1, 1, -1, 1 \ldots$

The range of this sequence is $\{1, -1\}$

4. The sequence $\{(-1)^{n+1}\}$ is given by $1, -1, 1, -1.$

The range of the sequence is also $\{1, -1\}$

The sequence $(-1)^n$ and $((-1)^{n+1})$ are different

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... sequence starts with -1, and the second sequence starts with 1.

5. The constant function $f: N \to R$ given by $f(n) = 1$ determines the sequences $1, 1, 1, \ldots\ldots$ Such a sequence is called a constant sequence.

6. The function $f: N \to R$ given by

$$f(n) = \begin{cases} \frac{1}{2} n & \text{if } n \text{ is even} \\ \frac{1}{2}(1-n) & \text{if } n \text{ is odd} \end{cases}$$

determine the sequence $0, 1, -1, 2, -2, \ldots, n, -n, \ldots$. The range of this sequence is Z.

7. The function $f: N \to R$ given by $f(n) = \frac{n}{n+1}$ determine the sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$

8. The function $f: N \to R$ given by $f(n) = \frac{1}{n}$ determine the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n}$

9. The function $f: N \to R$ given by $f(n) = 2n+3$ determines the sequences $5, 7, 9, 11, \ldots$

10. Let $x \in R$ The function $f: N \to R$ given by $f(n) = x^{n-1}$ determines the geometric sequences $1, x, x^2, \ldots, x^n, \ldots$

Definition:

A sequence can also be described by speciting the first few terms and starting a ~~rate~~ rule for determining an interms of the

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$a_1 = 1, \; a_2 = 1$ and $a_n = a_{n-1} + a$

, Then

$$a_3 = a_2 + a_1 = 2$$

$$a_4 = a_3 + a_2 = 3$$

$$\vdots \qquad \vdots$$

we obtain the sequences 1, 1, 2, 3, 5, 8, 13, ...

This sequence is called pibonaccis sequence.

write the first five terms of the following sequence.

i). $\frac{(-1)^n}{n!}$

soln::

$\frac{(-1)^n}{n}$, $n = 1, 2, 3, 4, 5$

$\Rightarrow n = 1 \Rightarrow \frac{(-1)^1}{1} = -1$; $n = 4 = \frac{(-1)^4}{4} = 1/4$

$n = 2 \Rightarrow \frac{(-1)^2}{2} = \frac{1}{2}$; $n = 5 = \frac{(-1)^5}{5} = -1/5$

$n = 3 \Rightarrow \frac{(-1)^3}{3} = \frac{-1}{3}$.

$\therefore -1, \; 1/2, \; -1/3, \; 1/4, \; -1/5$.

ii $\left[2/3 \left(1 - \frac{1}{10^n}\right) \right]$

soln:

$n = 1, 2, 3, 4, 5$

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(5) iii) $a_1 = 1$ and $a_{n+1} = \sqrt{(2+a_n)}$

soln: $n = 1, 2, 3, 4, 5.$

$n=1$

$$a_{1+1} = \sqrt{(2+a_1)} = \sqrt{2+1} = \sqrt{3} = 1.732$$

$n=2$

$$a_{2+1} = \sqrt{2+a_2} = \sqrt{2+1.732} = \sqrt{3.732} = 1.932$$

$n=3$

$$a_{3+1} = \sqrt{2+1.932} = \sqrt{3.932} = 1.983$$

$n=4$

$$a_{4+1} = \sqrt{2+1.983} = \sqrt{3.983} = 1.996$$

$n=5$

$$a_{5+1} = \sqrt{2+1.996} = \sqrt{3.996} = 1.999.$$

Bounded Sequences.

A sequence an is said to be bounded above if there exists a real number K such that $a_n \leq K$ for all n.

The K is called an upper bound of the sequence (an)

A sequences (an) is said to be bounded below if there exists a real number K such that $a_n \geq K$ for all n.

Then K is called a lower bound the sequence (an).

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$n=1$

$$\Rightarrow \left[\frac{2}{3}\left(1-\frac{1}{10^1}\right)\right]$$

$$= \frac{2}{3}\left(\frac{9}{10}\right)$$

$$= \frac{3}{5}$$

$n=2$

$$\Rightarrow \frac{2}{3}\left(1-\frac{1}{10^2}\right)$$

$$= \frac{2}{3}\left(1-\frac{1}{100}\right)$$

$$= \frac{2}{3}\left(\frac{99}{100}\right)$$

$$= \frac{33}{50}$$

$n=3$

$$\Rightarrow \frac{2}{3}\left(1-\frac{1}{1000}\right)$$

$$= \frac{2}{3}\left(\frac{999}{1000}\right)$$

$$= \frac{333}{500}$$

$n=4$

$$= \left[\frac{2}{3}\left(1-\frac{1}{10000}\right)\right]$$

$$= \frac{2}{3}\left(\frac{9999}{10000}\right)$$

$$= \frac{3333}{5000}$$

$n=5$

$$= \frac{2}{3}\left(1-\frac{1}{100000}\right)$$

$$= \frac{2}{3}\left(\frac{99999}{100000}\right)$$

$$= \frac{33333}{50000}$$

$\therefore$ The sequence: $3/5, \frac{33}{50}, \frac{333}{500}, \frac{3333}{5000}, \frac{33333}{50,000}$.

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A sequence (a_n) is said to be a bounded sequence if it is both bounded above and below.

Ex:-

1. Consider the sequence $1, 1/2, 1/3, \dots, 1/n$

Here 1 is the l.u.b and 0 is the g.l.b

It is a bounded sequence.

2. The sequence $1, 2, 3 \dots n \dots$ is bounded below but not bounded above 1 is the g.l.b. of the sequence.

3. The sequence $-1, -2, -3 \dots -n \dots$ is bounded above but not bounded below -1 is the l.u.b of the sequence.

4. $1, -1, 1, -1, \dots$ is a bounded sequence 1 is the l.u.b and -1 is the g.l.b of the sequence.

5. Any constant sequence is bounded Here l.u.b = g.l.b = The constant term sequence.

Find g.l.b and l.u.b of the following sequence.

a). $2, -2, 1, -1, 1, -1, \dots$

l.u.b = 2

g.l.b = -2

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b). 1, $\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{3}}$, $\frac{1}{\sqrt{4}}$, ... $\frac{1}{\sqrt{n}}$ ....

l.u.b = 1

c). 1/2, 2/3, 3/4 .... $\frac{n}{n+1}$ ....

l.u.b = 1/2

Monotonic sequence :-

A sequence (a_n) is said to be monotonic increasing if $a_n \leq a_{n+1}$ $\forall n$

A sequence a_n is said to be monotonic decreasing if $a_n \geq a_{n+1}$ $\forall n$

The sequence a_n is said to be strictly monotonic increasing if $a_n < a_{n+1}$ $\forall n$ strictly monotonic decreasing $a_n > a_{n+1}$ $\forall n$

The sequence (a_n) is ~~to~~ said to be monotonic sequence if it is either monotonic increasing or decreasing.

Ex :-

1. 1, 2, 2, 3, 3, 3, 4, 4, 4 is a monotonic increasing sequence.

2. 1, 2, 3, 4 .... n is a strictly monotonic increasing sequence.

3. 1, 1/2, 1/3, 1/4 .... $\frac{1}{n}$ is a strictly monotonic decreasing ~~de~~ sequence.

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4. 1, -1, 1, -1, ... is rather monotonic

(c) increasing for monotonic decreasing

Hence a_n is not monotonic sequence.

1. Solve that $\frac{2n-7}{3n+2}$ is a monotonic increasing sequence.

Soln:

Let, $a_n = \frac{2n-7}{3n+2}$

$$a_{n+1} = \frac{2(n+1)-7}{3(n+1)+2}$$

$$= \frac{2n+2-7}{3n+3+2}$$

$$= \frac{2n-5}{3n+5}$$

$$a_n - a_{n+1} = \frac{2n-7}{3n+2} - \frac{2n-5}{3n+5}$$

$$= \frac{(3n+5)(2n-7) - (2n-5)(3n+2)}{(3n+2)(3n+5)}$$

$$= \frac{6n^2 - 21n + 10n - 35 - 6n^2 + 15n - 4n + 10}{(3n+2)(3n+5)}$$

$$= \frac{-25}{(3n+2)(3n+5)} < 0$$

$a_n - a_{n+1} < 0$

$a_n < a_{n+1}$

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9 Hence a_n is a monotonic sequence increasing sequence.

Shoco.

2. show that a_n is a monotonic sequence the $\left(\frac{a_1+a_2+\ldots\ldots+a_n}{2}\right)$ is also monotonic sequence.

Soln:

Given a_n is a monotonic sequence.

Let a_n is a monotonic increasing sequence.

Then

$$a_1 \le a_2 \le 2\ a_3 \le \ldots\ldots \le a_n \longrightarrow (1)$$

$$b_n = \left(\frac{a_1+a_2+\ldots.+a_n}{n}\right)$$

$$b_{n+1} = \left(\frac{a_1+a_2+\ldots.+a_{n+1}}{n+1}\right)$$

$$b_{n+1} - b_n = \left(\frac{a_1+a_2+\ldots.+a_{n+1}}{n+1}\right) - \left(\frac{a_1+a_2+\ldots+a_n}{n}\right)$$

$$= \frac{n\,(a_1+a_2+\ldots\ldots+a_{n+1}) - (n+1)(a_1+a_2+\ldots.+a_n)}{n(n+1)}$$

$$= \frac{na_1+na_2+\ldots\ na_{n+1} - na_1 + na_2 - na_n - (a_1+a_2+\ldots.+a_n)}{n(n+1)}$$

$$= \frac{na_{n+1} - na_n - (a_1+a_2+\ldots.+a_n)}{n(n+1)}$$

$$\frac{n(a_{n+1} - a_n)}{n(n+1)} \ge 0$$

$$b_{n+1} - b_n \ge 0$$

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b_n is a monotonic increasing sequence.

$\dfrac{a_1+a_2+\dots+a_n}{n}$ is a monotonic increasing sequence

Convergent sequence:-

A sequence (a_n) is said to converge to a number l, if given $\epsilon>0$ there exists a positive integer m such that $|a_n-l|<\epsilon\ \forall\ n\geq m$. We ~~stay~~ say that l is the limit of the sequence and we write $\lim_{n\to\infty} a_n = l$ (or) $a_n\to l$.

Theorem:-

A sequence cannot converge to two different limits.

Proof:

Let (a_n) be a convergent sequence if possible let l_1 and l_2 be two distinct limits of (a_n)

let $\epsilon>0$ be given

since $(a_n)\to l_1$ There exists a natural numbers n_1 such that $|a_n-l_1|<\frac{1}{2}\epsilon\ \forall\ n\geq n_1$ $\to(1)$

since $(a_n)\to l_2$. There exists a natural number n_2 suchthat $|a_n-l_2|<\frac{1}{2}\epsilon\ \forall\ n\geq n_2$ $\to(2)$

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Let $m = \max\{n_1, n_2\}$

Then $|l_1 - l_2| = |l_1 - a_m + a_m - l_2|$

$\leq |a_m - l_1| + |a_m - l_2|$

$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon$ [by (1) and (2)]

$= \varepsilon$

$\therefore |l_1 - l_2| < \epsilon$ for every $\epsilon > 0$.

clearly $l_1 - l_2 = 0$.

$\therefore l_1 = l_2$

Examples :

1. $\lim\limits_{n \to \infty} \frac{1}{n} = 0$ (or) $(1/n) \to 0$

proof:

Let $\varepsilon > 0$ be given

Then $|\frac{1}{n} - 0| = \frac{1}{n} < \epsilon$ in $n > 1/\epsilon$

Hence if we choose m to be any natural number such that $m > 1/\epsilon$

Then $|\frac{1}{n} - 0| < \epsilon \ \forall \ n \geq m$

$\therefore \lim\limits_{n \to \infty} \frac{1}{n} = 0$.

2. The constant sequence 1, 1, 1 .... converge ~~to~~ of 1.

proof.

Let $\epsilon > 0$ be given

Let the given sequence be denoted by (a_n)

Then $a_n = 1 \ \forall n$

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$|a_n - 1| = |1-1| = 0 < \epsilon \quad \forall\, n \in \mathbb{N}$

$\therefore |a_n - 1| < \epsilon \quad \forall\, n \geq m$ where m can be chosen to be any natural number.

$$\therefore \lim_{n \to \infty} a_n = 1.$$

3. $$\lim_{n \to \infty} \frac{n+1}{n} = 1$$

sol

$$a_n = \frac{n+1}{n}$$

$$= n/n + 1/n$$

$$a_n = 1 + \frac{1}{n}$$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} (1 + 1/n)$$

$$= 1 + 1/\infty$$

$$= 1 + 0$$

$$= 1$$

By definition $|a_n - l| < \epsilon$

$$\left| \frac{n+1}{n} - 1 \right| < \epsilon$$

$$\Rightarrow |1 + 1/n - 1| \leq \epsilon$$

$$|1/n| < \epsilon.$$

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13 4. $\lim_{n\to\infty} 1/2^n = 0.$

proof: Let $\epsilon > 0$ be given

Then $\left|\frac{1}{2^n} - 0\right| = \frac{1}{2^n} < \frac{1}{n}$ [$\because 2^n > n$ for all n]

$\therefore \left|\frac{1}{2^n} - 0\right| < \epsilon \;\forall\; n \geq m$ where m is any,

natural number greater than $1/\epsilon$

$\therefore \lim_{n\to\infty} \frac{1}{2^n} = 0.$

5. The sequence $((-1)^n)$ is not convergent

proof:

support the sequence $(-1)^n$ convergent to l.

Then given $\epsilon > 0$. There exists a natural number m such that

$|(-1)^n - l| < \epsilon \quad \forall\; n \geq m$

$\therefore |(-1)^m - (-1)^{m+1}| = |(-1)^m - l + l - (-1)^{m+1}|$

$\leq |(-1)^m - l| + |(-1)^{m+1} - l|$

$< \epsilon + \epsilon$

$= 2\epsilon$

But $|(-1)^m - (-1)^{m+1}| = 2$

$\therefore 2 < 2\epsilon$ i.e., $1 < \epsilon$

which is a contradiction since $\epsilon > 0$ is arbitrary.

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$\therefore$ The sequence $((-1)^n)$ is not convergent.

3. Any convergent sequence is a bounded sequence.

Proof

Let (a_n) be a convergent sequence.

Let $\lim_{n \to \infty} a_n = l$

Let $\epsilon > 0$ be given. Then there exists $m \in n$ such that $|a_n - l| < \epsilon \ \forall \ n \geq m$.

$\therefore |a_n| < |l| + \epsilon \ \forall \ n \geq m$

Now, Let $k = \max \{|a_1|, |a_2| \dots |a_{m-1}|, |l| + \epsilon\}$

Then $|a_n| \leq k \ \forall n$

$\therefore a_n$ is a bounded sequence.

Divergent sequence:

A sequence (a_n) is said to diverge to ∞ if given any real number $k > 0$. There exists $m \in N$ such that $a_n \geq k \ \forall \ n \geq m$

In symbols we write

$a_n \to \infty$ (or) $\lim_{n \to \infty} a_n = \infty$

A sequence (a_n) is said to diverge to $-\infty$ if given any real number $k < 0$. There exists $m \in N$ such that $a_n < k \ \forall \ n \geq m$

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In symbols $\lim_{n\to\infty} a_n = -\infty$ (or) $(a_n) \to -\infty$.

Ex:-

1. $(n^2) \to \infty$.

proof:

Let $k > 0$ be any given real number.

choose m to be any natural number such that $m > \sqrt{k}$.

Then $n^2 > k$ for all $n \geq m$

$\therefore (n^2) \to \infty$.

2. $(2^n) \to \infty$.

proof:

Let $k > 0$ be any given real number.

Then $2^n > k \iff n \log 2 > \log k$.

$\iff n > (\log k) / \log 2$.

Hence if we choose m, to be any natural number $m > (\log k) / \log 2$. then $2^n > k$ for all $n \geq m$.

$\therefore (2^n) \to \infty$.

Theorem 3.3:-

$(a_n) \to \infty$. iff $(-a_n) \to -\infty$.

proof:

Let $(a_n) \to \infty$

Let $k < 0$ be any given real number. since $(a_n) \to \infty$ there exists $m \in N$ such that $a_n > -k$ for all $n \geq m$.

$\therefore -a_n < k$ for all $n \geq m$.

$\therefore (-a_n) \to -\infty$.

$|||^{ly}$ we can prove that if $(-a_n) \to -\infty$ then $(a_n) \to \infty$.

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Theorem 3.4

If $(a_n) \to \infty$ and $a_n > 0$ for all $n \in N$ then $(1/a_n) \to 0$.

Proof:

Let $\varepsilon > 0$ be given. Since $(a_n) \to \infty$, there exists $m \in N$ such that $a_n > 1/\varepsilon$ for all $n \geq m$.

$\therefore \frac{1}{a_n} < \varepsilon$ for all $n \geq m$.

$\therefore \left|\frac{1}{a_n}\right| < \varepsilon$ for all $n \geq m$.

$\therefore (1/a_n) \to 0$.

Note:

The converse of the above theorem is not true. For example consider the sequence (a_n) where $a_n = \frac{(-1)^n}{n}$. Clearly $(a_n) \to 0$.

Now $\left(\frac{1}{a_n}\right) = \left(\frac{n}{(-1)^n}\right) = -1, 2, -3, 4 \ldots$ which neither nor diverges to ∞ or $-\infty$.

Thus if a sequence $(a_n) \to 0$, then the sequence $(1/a_n)$ converge or diverge.

Theorem 3.5

If $(a_n) \to 0$ and $a_n > 0$ for all $n \in N$, then $(1/a_n)$.

Proof:

Let $k > 0$ be any given real number. Since $(a_n) \to 0$ $m \in N$ such that $|a_n| < 1/k$ for all $n \geq m$.

$\therefore a_n < 1/k$ for all $n \geq m$ (since $a_n > 0$).

$\therefore 1/a_n > k$ for all $n \geq m$.

$\therefore (1/a_n) \to \infty$.

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Theorem 3.6.

Any sequence (a_n) diverging to ∞ is bounded below & bounded above.

proof.

Let $(a_n) \to \infty$. Then for any given real number $k > 0$

then $m \in N$ such that $a_n > k$ for all $n \geq m$.

$\therefore$ K is not an upper bound of the sequence (a_n)

$\therefore$ (a_n) is not bounded above.

Now let $l = \min (a_1, a_2, \ldots, a_m, k\}$.

from (1) we see that $a_n \geq \lambda$ for all n.

$\therefore$ (a_n) is bounded below.

Theorem 3.7

Any sequence (a_n) diverging to $-\infty$ is bounded below.

proof: is similar to that of Theorem 3.6.

Definition:

A sequence (a_n) is said to diverge to $-\infty$ if given $K < 0$ there exists $m \in N$ such that $a_n < k$ for all $n \geq m$. In should $\lim\limits_{n \to \infty} a_n = -\infty$ (or) $(a_n) \to -\infty$.

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