LESSON 7

GRAPH THEORY

A *linear graph* (or simply a graph) $G = (V,E)$ cosists of a set of objects

$V = \{v_1,v_2,\ldots\}$ called *vertices*, and another set $E = \{e_1,e_2,\ldots\}$, whose elements are called *edges*, such that each edge e_k is identified with an unordered pair (v_i,v_j) of vertices.

The vertices vi, vj associated with edge ek are called the end vertices of ek.

The most common representation of a graph is by means of a diagram, in which the vertices are represented as points and each edge as a line segment joining its end vertices.

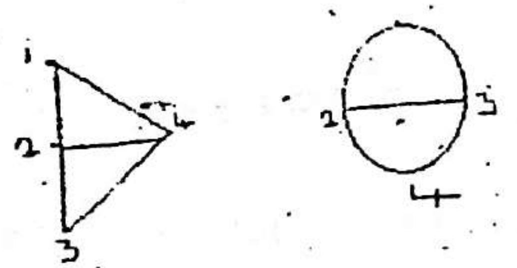
An edge having the same vertex as both its end vertices is called a *self-loop*. The above figure is a self loop. Also more than one edge associated with a given pair of vertices. Such edges are referred to as *parallel edges.*

A graph that has neither self-loops nor parallel edges is called a *simple graph*..

It should be noted that, in drawing a graph, it is immaterial whether the lines are drawn straight or curved, long or short: what is important is the incidence between the edges and vertices.

Fig (1): Same graph drawn differently.

loop - itself



A graph is also called a *linear complex*, a *1-comple.*
or a *one-dimensional* complex. A vertex is also referred t
as a *node*, a *junction*, a *point*, *0-cell*, or an 0-simplex. Othe
terms used for an edge are a branch, a line, an element, a 1
cell, an arc and a 1-simplex.

A graph with finite number of vertices as well a
finite number of edges is called a *finite graph*; otherwise, i
is an *infinite graph*.

Applications of Graphs:

Graph theory has a very wide range of application
in engineering, in physical, social, and biological sciences
in linguistics, and in numerous other areas. A graph can b
used to represent almost any physical situation involving
discrete objects and a relationship among them.

Incidence and Degree

When a vertex v_i is an end vertex of some edge e_j, v
and e_j are said to be incident with to each other. Two non-
parallel edges are said to be adjacent if they are incident on
a common vertex. Two vertices are said to be adjacent if
they are the end vertices of the same edge.

The number of edges incident on a vertex vi, with
self-loops counted twice, is called the degree, $d(v_i)$ of the

vertex v_i. The degree of a vertex is sometimes also referred to as its valency.

2m

Let G be a graph with e edges and n vertices $v_1, v_2, \ldots v_n$. Since each edge contributes two degrees, the sum of the degrees of all vertices in G is twice the number of edges in G. That is,

$$\sum_{i=1}^{n} d(v_i) = 2e \qquad \text{----(1)}$$

Theorem.: The number of vertices of odd degree in a graph is always even.

Proof:

If we consider the vertices with odd and even degrees separately, the quantity in the left side of Eq(1) can be expressed as the sum of two sums, each taken over vertices of even and odd degrees, respectively as follows:

$$\sum_{i=1}^{n} d(v_i) = \sum_{\text{even}} d(v_j) + \sum_{\text{odd}} d(v_k) \qquad \text{----(2)}$$

Since the left-hand side in Eq (2) is even, and the first expression on the right-hand side is even(being a sum of even numbers), the second expression must also be even:

$$\sum_{\text{odd}} d(v_k) = \text{an even number} \qquad \text{---(3)}$$

Because in Eq(3) each $d(v_k)$ is odd, the total number of terms in the sum must be even to make the sum an even number. Hence the theorem.

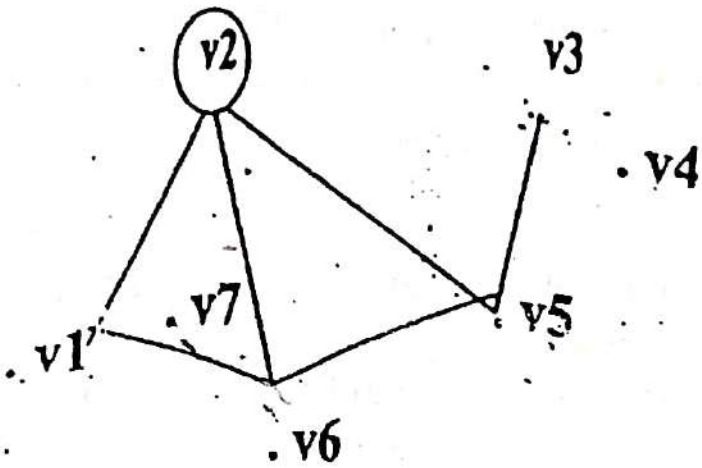
Regular graph:

A graph in which all vertices are of equal degree is called a **regular graph.**

Isolated vertex, Pendant vertex, and null graph

A vertex having no incident edge is called an *isolated vertex.* In other words, isolated vertices are vertices with *zero degree.* A vertex of degree one is called a *pendant vertex* or an end vertex. Two adjacent edges are said to be in *series* if their common vertex is of degree two.

Fig:2 Graph containing isolated vertices, series edges, and a pendant vertex.



Vertices v4 and v7 in the above example are called isolated vertices.

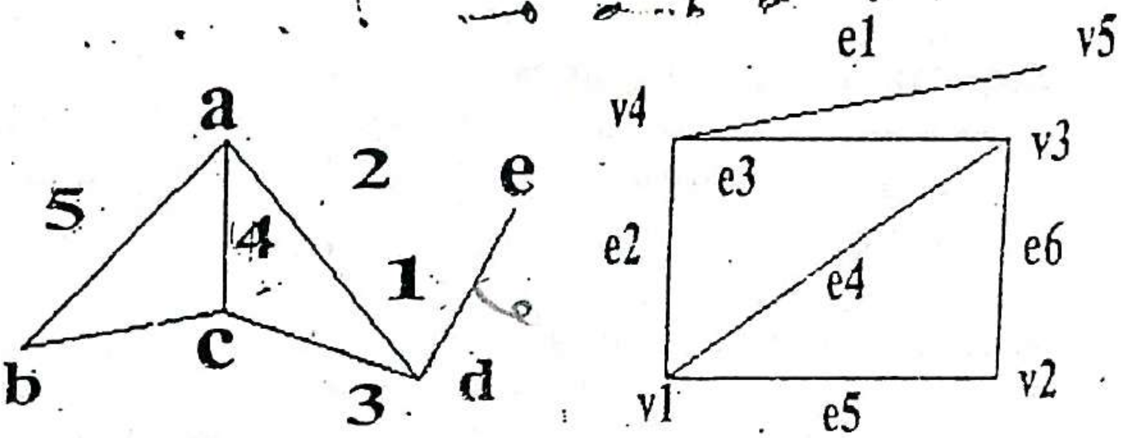
Vertex v3 is called pendant vertex.

Two edges incident on v1 are in series.

Null graph:- A graph without any edges is called a null graph. In other words, every vertex in a null graph is an isolated vertex.

Isomorphism:- Two graphs G and G' are said to be isomorphic if there is a one-to-one correspondence between their vertices and between their edges such that the incidence relationship is preserved.

Fig 3:- Isomorphic graphs



By the definition of isomorphism that two isomorphic graphs must have

1. the same number of vertices
2. the same number of edges.
3. an equal number of vertices with a given degree.

Subgraphs:- A graph g is said to be a subgraph of a graph G if all the vertices and all the edges of g are in G, and each edge of g has the same end vertices in g as in G. A subgraph can be thought of as being contained in another graph. The following observations can be made :

1) Every graph is its own subgraph.

2) A subgraph of a subgraph of G is a subgraph of G

3) A single vertex in a graph G is a subgraph of G

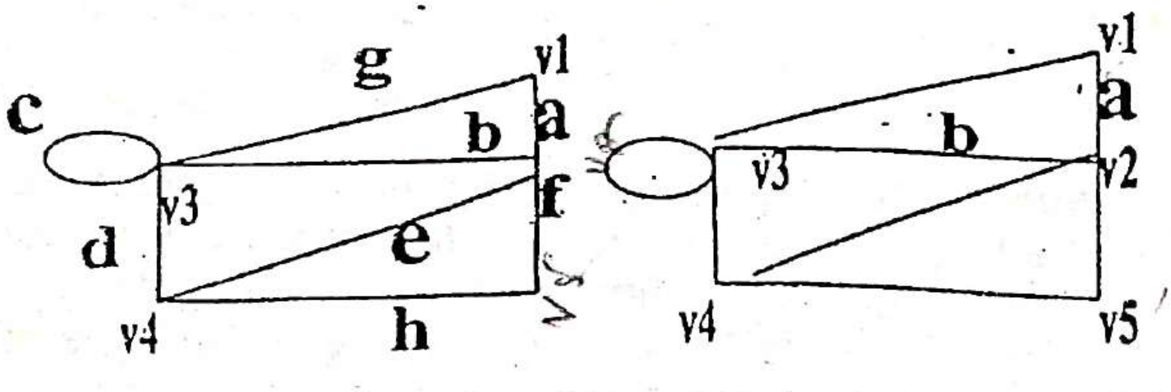
4) A single edge in G, together with its end vertices, is also a subgraph of G

Edge-Disjoint Subgraphs: Two (or more) graphs g_1 and g_2 of a graph G are said to be edge disjoint if g_1 and g_2 do not have any edges in common. Subgraphs that do not even have vertices in common are said to be vertex disjoint.

Walks, Paths and Circuits

A walk is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident with vertices preceding and following it. No edge appears more than once in a walk. A vertex however may appear more than once.

Fig 4:- A walk and a path



A *walk* is also referred to as an edge train or a chain. Vertices with which a walk begins and ends are called its terminal vertices. It is possible for a walk to begin and end at the same vertex. Such a walk is called a *closed walk*. A walk that is not closed is called an *open walk*.

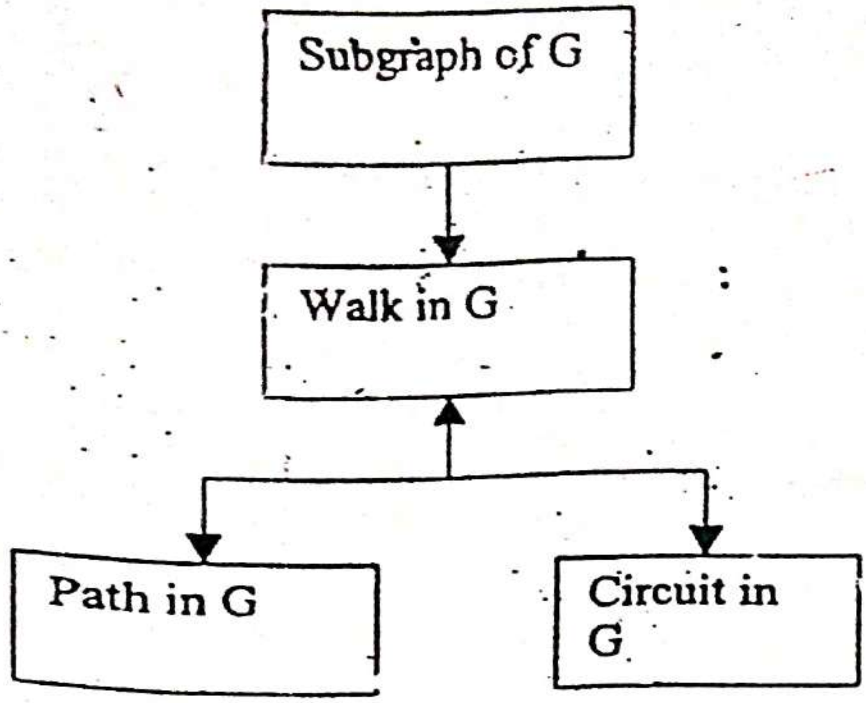
An open walk in which no vertex appears more than once is called a *path*. The number of edges in a path is called the *length of a path*.

The terminal vertices of a path are of degree one, and the rest of the vertices are of degree two. The degree is counted only with respect to the edges included in the path and not the entire graph in which the path may be contained.

A closed walk in which no vertex appears more than once is called a *circuit*. That is, a circuit is a closed, non-intersecting walk. A circuit is also called a cycle, elementary cycle, circular path, and polygon.

Vertices v1 and v5 are the terminal vertices of of the walk in the fig 4
V1 av2 bv3 d v4 is a path in the above figure.

Figure 5:- Walks, paths and circuits as subgraphs

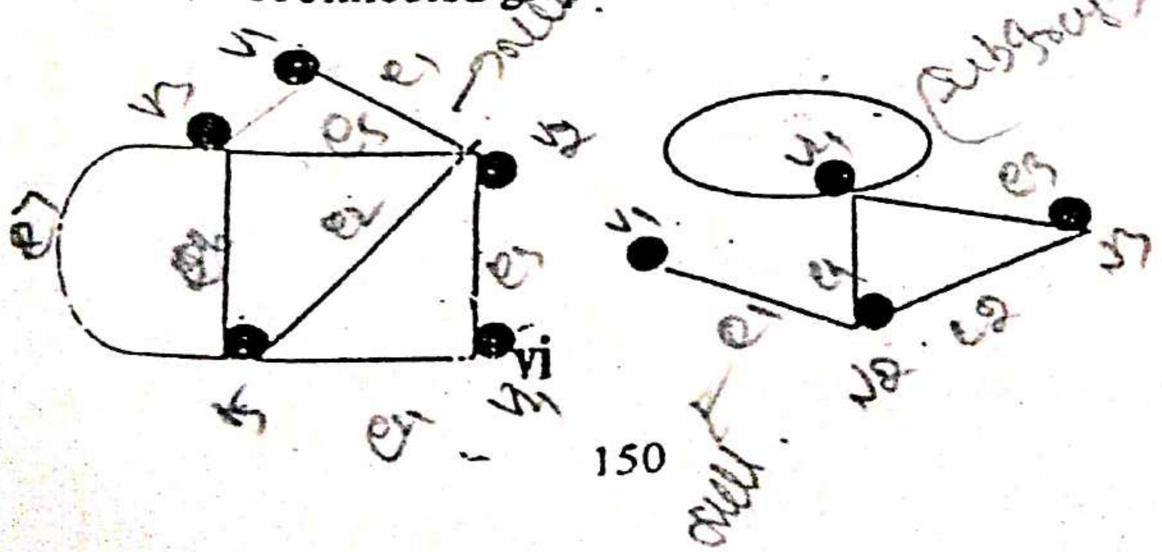


Connected Graphs, Disconnected Graphs and Components

A graph is connected if we can reach any vertex from any other vertex by traveling along the edges. More formally:

A Graph G is said to be connected if there is atleast one path between every pair of vertices in G. Otherwise, G is disconnected. A null graph of more than one vertex is disconnected. Each of these connected subgraphs is called a component.

Fig 6:- A disconnected graph with two components



The above fig(6) consist ; of two components. Another way of looking at the component is as follows: Consider a vertex v_i in a disconnected graph G. By definition, not all vertices of G are joined by paths to v_i. Vertex vi and all the vertices of G that have paths to v_i, together with all the edges incident on them, form a component. Obviously, a component itself is a graph.

Theorem :- A graph G is disconnected if and only if its vertex set V can be partitioned into two nonempty, disjoint subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in subset V_1 and the other in subset V_2.

Proof:

Suppose that such a partitioning exists. Consider two artitrary vertices a and b of G, such that $a \varepsilon V_1$ and $b \varepsilon V_2$. No path can exist betweem vertices a and b; Otherwise, there would be atleast one edge whose one end vertex would be in V_1 and the other in V_2. Hence, if a partition exists, G is not connected.

Conversely, let G be a disconnect graph. Consider a vertex a in G. Let V_1 be the set of all vertices that are joined by paths to a. Since G is disconnected, V_1 does not include all vertices of G. The remaining vertices will form a set V_2. No vertex in V_1 is joined to any in V_2 by and edge. Hence the partition.

Theorem

If a graph (connected or disconnected) has exactly two vertices of odd degree, there must be a path joining these two vertices.

Proof

Let G be a graph with all even verticest except vertices V_1 and V_2, which are odd. Therefore for every component of a disconnected graph, no graph can have an odd number of odd vertices. Therefore, in graph G, V_1 and V_2 must belong to the same component, and hence must have a path between them.

Theorem

A simple graph with n vertices and k components can have at most $(n-k)(n-k+1)/2$ edges.

Proof:

Let the number of vertices in each of the k components of a graph G be $n_1, n_2, \ldots, n_k$. Thus we have

$$n_1 + n_2 + \ldots + n_k = n,$$

$$n_i \geq 1.$$

$$\sum_{i=1}^{k} n_i^2 \leq n^2 - (k-1)(2n-k)$$

Now the maximum number of edges in the ith component of G is $(1/2)\, n_i(n_i-1)$. Therefore the maximum number of edges in G is

$$\frac{1}{2}\sum_{i=1}^{k}(n_i-1)\,n_i = \frac{1}{2}\left(\sum_{i=1}^{k} n_i^2\right) - \frac{n}{2}$$

$$\leq \frac{1}{2}\left[n2-(k-1)(2n-k)\right] - \frac{n}{2}$$

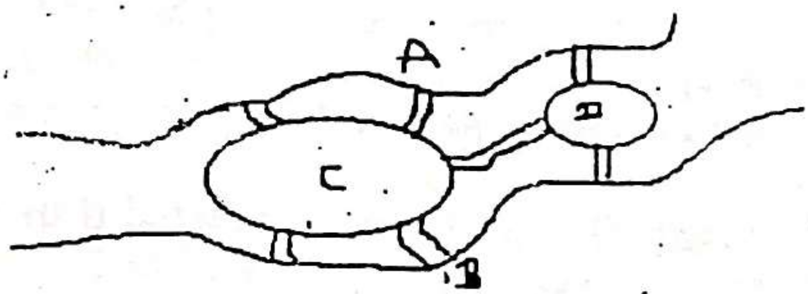
$$= (1/2)(n-k)(n-k+1).$$

Konigsberg Bridge Problem: The konigsberg bridge problem is perhaps the best-known example in graph theory

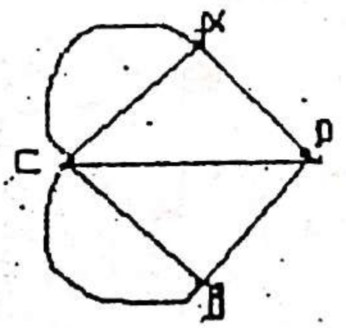
It was a long-standing problem until solved by Leonhard Euler (1707-1783) in 1736, by means of a graph. Euler wrote the first paper ever in graph theory and thus became the originator of the theory of graphs.

Two islands, C and D, formed by the Pregel River in Konigsberg were connected to each other and to the banks A and B with seven bridges. The problem was to start at any of the four land areas of the city, A,B, C or D walk over each of the seven bridges *exactly* once and return to the starting point

Fig 7: Kongisberg bridge problem



Euler represented this situation by means of a graph as shown in the following figure,
Fig 8.



... :- If some closed walk in a graph contains al the edges of the graph, then the walk is called an *Euler lin* and the graph an Euler graph. And , an euler graph i always connected.

Theorem :-

A given connected graph G is an Euler graph if an only if all vertices of G are of even degree.

Proof:-

Suppose that G is an Euler graph. It therefor contains an Euler line (which is a closed walk). In tracin this walk we observe that every time the walk meets a vertex v it goes through two "new edges incident on v - with one we "entered" v and with other "exited". This is true not only of all intermediate vertices of the walk but also of the terminal vertex, because we "exited" and "entered" the same vertex at the beginning and end of the walk, respectively. Thus if G is an Euler graph, the degree of every vertex is even.

To prove the sufficiency of the condition, assume that all vertices of G are of even degree. Now we construct a walk starting at an arbitrary vertex v and going through the edges of G such that no edge is traced more than once. We continue tracing as far as possible. Since every vertex is of even degree, we can exit from every vertex we enter; the tracing cannot stop at any vertex but v. And since v is also of even degree, we shall eventually reach v when the tracing comes to an end. If this closed walk h we just traced includes all the edges of G, G is an Euler graph. If not, we remove from G all the edges in h and obtain a subgraph h' of G formed by the remaining edges. Since both G and h have all their vertices of even degree, the degree of the vertices of h' are also even. Moreover, h' must touch h at least at one vertex a, because G is connected. Starting from a, we can again construct a new walk in graph h'. Since all the vertices of h' are of even degree, this walk in h' must terminate at vertex a; but this walk in h' can be combined with h to form a new walk, which starts and ends at vertex v and has more edges than h. This process can be repeated until we obtain a closed walk that traverses all the edges of G. Thus G is an Euler graph.

Unicursal Graph :-

A open walk that includes all edges of a graph without retracing any edge is a unicursal line or an open ~~Euler~~ line. A connected that has a unicursal line will be called a unicursal graph.

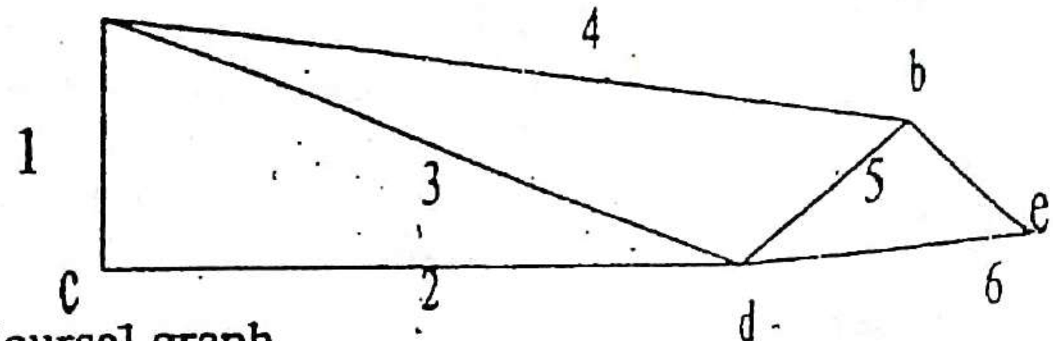


Fig 9: Unicursal graph

Theorem :-

In a connected graph G with exactly 2k odd vertices, there exist k edge-disjoint subgraphs such that they together contain all edges of G and that each is unicursal graph.

Proof :-

Let the odd vertices of the given graph G be named v1, v2..., vk; w1,w2,....wk in any arbitrary order.

Add k edges to G, between the vertex pairs (v_1,w_1), (v_2,w_2),....,(v_k, w_k) to form a new graph G'.

Since every vertex of G' is of even degree, G consists of an Euler line ρ. Now if we remove from ρ the k edges we just added, ρ will be split into k walks, each of which is a unicursal line. The first removal will leave a single unicursal line. The second removal will split that into two unicursal lines; and each successive removal will split a unicursal line into two unicursal lines, until there are k of them. Thus the theorem.

Cut set:-

In a connected graph G, a cut-set is a set of edges whose removal from G leaves G disconnected, provided removal of no proper subset of these edges disconnects G. Sometimes a cut-set is also called a cocycle.. A cut-set always "cuts" a graph into two. Therefore, a cut-set can also be defined as a minimal set of edges in a connected graph whose removal reduces the rank of the graph by one.

Edge connectivity:

Each cut-set of a connect graph G consists of a certain number of edges. The number of edges in the smallest cut-set is defined as the edge connectivity of G. Equivalently, the edge connectivity of a connected graph can be defined as the minimum number of edges whose removal reduces the rank of the graph by one.

Vertex connectivity:- The vertex connectivity of a connected graph G is defined as the minimum number of vertices whose removal from G leaves the remaining graph disconnected.

Separable Graph: A connected graph is said to be separable if its vertex connectivity is one. All other connected graphs are nonseparable.

Theorem: The edge connectivity of a graph G cannot exceed the degree of the vertex with the smallest degree in G

Proof:

Let vertex v_i be the vertex with the smallest degree in G. Let $d(v_i)$ be the degree of v_i. Vertex v_i can be separated from G by removing the $d(v_i)$ edges incident on vertex v_i. Hence the theorem

Incident Matrix:

Let G be a graph with n vertices, e edges, and no self-loops. Define an n by e matrix $A = [a_{ij}]$, whose n rows correspond to the n vertices and the e column correspond to the edges, as follows:

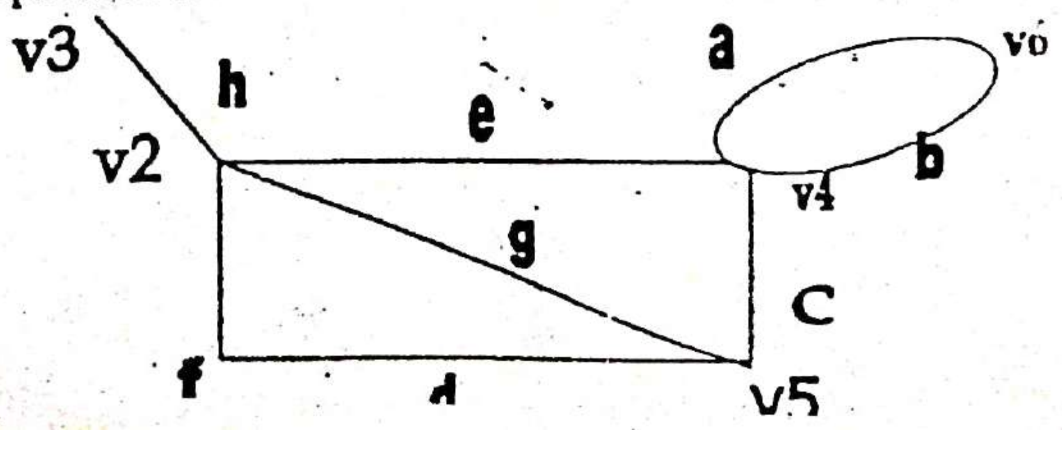
The matrix element

$a_{ij} = 1$, if the jth edge e_j is incident on ith vertex v_i, and

$= 0$, otherwise

Such a matrix A is called the vertex-edge incidence matrix, or simply *incidence matrix*. Matrix A for a graph G is sometimes also written as A(G).

Fig:- Graph and its incidence matrix



$$
\begin{array}{c c}
 & \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\
\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} &
\left(\begin{array}{cccccccc}
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 10 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
\end{array}
$$

The incidence matrix contains only two elements 0 and 1. I is also called a binary matrix or a (0-1) matrix. The following observations about the incidence matrix A car readily be made:

1. Since every edge is incident on exactly two vertices, each column of A has exactly two 1's.

2. The number of 1's in each row equals the degree of the corresponding vertex.

3. a row with all 0's, therefore, represents an isolated vertex.

4. Parallel edges in a graph produce identical columns in its incidence matrix

5. If a graph G is disconnected and consists of two components g_1 and g_2, the incidence matrix A(G) of graph G can be written in a block diagonal form as

$$
A(G) = \left[\begin{array}{c|c} A(g_1) & 0 \\ \hline 0 & A(g_2) \end{array}\right]
$$

Where $A(g_1)$ and $A(g_2)$ are the incidence matrices of components g_1 and g_2.

6. Permutation of any two rows or columns in an incidence matrix simply corresponds to relabelling the vertices and edges of the same graph.

Rank of the Incidence Matrix; Each row in an incidence matrix A(G) may be regarded as a vector over GF(2) in the vector space of graph G. Let the vector in the first row be called A_1, in the second row A_2, and so on. Thus

$$A(G) = \begin{pmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{pmatrix}$$

Since there are exactly two 1's in every column of A, the sum of all these vectors is 0. Thus vectors $A_1, A_2, \ldots, A_n$ are not linearly independent. Therefore, the rank of A is less than n; that is, rank $A \leq n - 1$.

Now consider the sum of any m of these n vectors ($m \leq n - 1$). If the graph is connected, A(G) cannot be partitioned, such that $A(g_1)$ is with m rows and $A(g_2)$ with n-m rows. In other words, no m by m submatrix of A(G) can be found, for $m \leq n - 1$, such that the modulo 2 sum of those m rows is equal to zero.

Since there are only two constants 0 and 1 in this field, the additions of all vectors taken m at a time for m =

1, 2,....., n − 1 exhausts all possible linear combinations o n − 1 row vector, Thus we have just shown that no linea combination of m row vecots of A can be equal to zerc Therefore, the rank of A(G) must be atleast n − 1.

Path Matrix :-

Another (0, 1) matrix often convenient to use i communication and transportation networks is the patl matrix. A path matrix is defined for a specific pair o vertices in a graph, say (x, y) and it is written as P (x, y). The rows in P(x, y) correspond to different paths betweer vertices x and y, and the columns correspond to the edge: in G. That is, the path matrix for (x, y) vertices is P(x, y) = $[p_{ij}]$, where

p_{ij} = 1, if jth edge lies in ith path, and

= 0, otherwise.

Some of the observations one can make at once about a path matrix P(x, y) of a Graph G are

1. A column of all 0's corresponds to an edge that does not lie in any path between x and y.
2. A column of all 1's corresponds to an edge that lies in every path between x and y.
3. There is no row with all 0's.
4. The ring sum of any two rows in P(x, y) corresponds to a circuit or an edge-disjoint union of circuits.

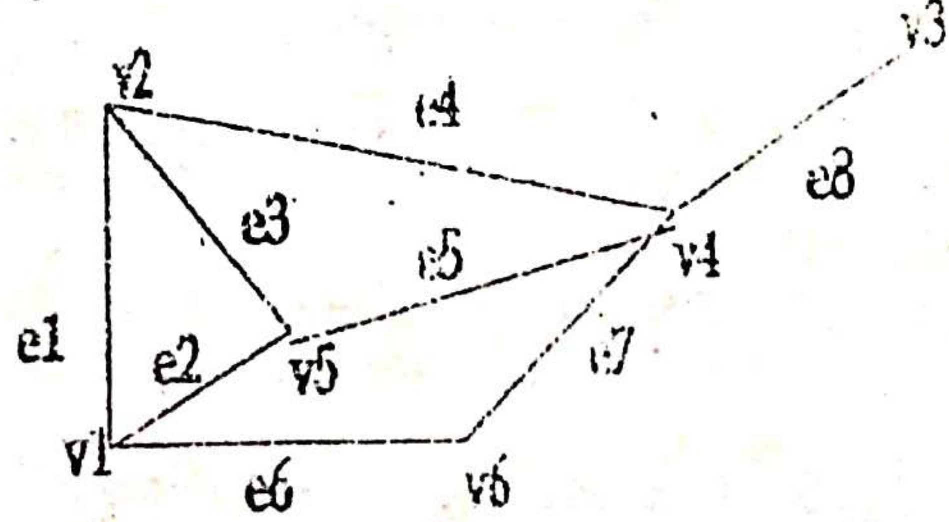
Ring Sum of two graphs G_1 and G_2 is a graph consisting of the vertex set $V_1 \cup V_2$ and of edges that are either in G_1 or G_2, but not in both.

Adjacency Matrix :-

The adjacency matrix of a graph G with n vertices and no parallel edges is an n by n symmetric binary matrix $X = [x_{ij}]$ defined over the ring of the integers such that

$x_{ij} = 1$, if there is an edge between ith and jth vertices, and
$= 0$, if there is no edge between them.

Fig: Simple graph and its adjacency matrix



	v1	v2	v3	v4	v5	v6
v_1	0	1	0	0	1	1
v_2	1	0	0	1	1	0
v_3	0	0	0	1	0	0
v_4	0	1	1	0	1	1
v_5	1	1	0	1	0	[illegible]
[illegible]	[illegible]	1	0	1	0	[illegible]