LESSON-3

ASSIGNMENT PROBLEM

INTRODUCTION

The assignment problem is a particular case of the transportation problem in which the objective is to assign a number of tasks(jobs or origins or sources) to an equal number of facilities(machines or persons or destinations) at a minimum cost(or a maximum profit).

Suppose that we have 'n' jobs to be performed on 'm machines(one job to one machine) and our objective is to assign the jobs to the machines at the minimum cost(or maximum profit) under the assumption that each machine can perform each job but with varying degree of efficiencies.

The assignment problem can be stated in the form of mxn matrix (c_{ij}) called a *Cost matrix* (or) *Effectiveness matrix* where c_{ij} is the cost of assigning i^{th} machine to the j^{th} job.

		Jobs				
		1	2	3	....	n
	1	c_{11}	c_{12}	c_{13}.	....	c_{1n}
	2	c_{21}	c_{22}	c_{23}	....	c_{2n}
Machines	3	c_{31}	c_{32}	c_{33}	....	c_{3n}
	:	...	...	...	....	...
	:	...	...	...	....	...
	:	...	...	...	....	...
	m	c_{m1}	c_{m2}	c_{m3}	....	c_{mn}

Mathematical formulation of an assignment problem:-

Consider an assignment problem of assigning n jobs to n machines(one job to one machine). Let c_{ij} be the unit cost of assigning i^{th} machine to the j^{th} job and

Let $x_{ij} = \begin{cases} 1, & \text{if } j^{th} \text{ job is assigned to } i^{th} \text{ machine} \\ 0, & \text{if } j^{th} \text{ job is not assigned to } i^{th} \text{ machine} \end{cases}$

The assignment model is given by the following LPP

$$\text{Minimize } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij}$$

subject to the constraints

$$\sum_{i=1}^{n} x_{ij} = 1, \qquad j = 1, 2, \ldots\ldots n$$

$$\sum_{j=1}^{n} x_{ij} = 1, \qquad i = 1, 2, \ldots\ldots n$$

and $x_{ij} = 0$ (or) 1.

The technique used for solving assignment problem makes use of the following two theorems:

Theorem 1: The optimum assignment schedule remains unaltered if we add or subtract a constant from all the elements of the row or column of the assignment cost matrix.

Theorem 2: If for an assignment problem all $c_{ij} > 0$, then an assignment schedule(x_{ij})

which satisfies $\sum_{i=1}^{n} c_{ij}\, x_{ij} = 0$ must be optimal.

Assignment Algorithm (or) Hungarian Method

First check whether the number of rows is equal to the number of columns. If it so, the assignment problem is said to be balanced. Then proceed to step1. If it is not balanced, then it should be balanced before applying the algorithm.

Step1:- Subtract the smallest cost element of each row from all the elements in the row of the given cost matrix. See that each row contains atleast one zero.

Step 2:- Subtract the smallest cost element of each column from all the elements in the column of the resulting cost matrix obtained by step 1.

Step 3:-(Assigning the Zeroes)

a) Examine the rows successively until a row with exactly one unmarked zero is found. Make an assignment to this single unmarked zero by encircling it. Cross all other zeroes in the column of this encircled zero, as these will not be considered for any future assignment. Continue in this way until all the rows have been examined.

b) Examine the columns successively until a column with exactly one unmarked zero is found. Make an assignment to this single unmarked zero by encircling it and cross any other zero in its row. Continue until all the columns have been examined.

Step 4:- (Apply optimal Test)

a) If each row and each column contain exactly one encircled zero, then the current assignment is optimal.

b) If atleast one row/column is without an assignment(i.e., if there is atleast one row/column is without one encircled zero), then the current assignment is not optimal. Go to step 5.

Step 5:- Cover all the zeroes by drawing a minimum number of straight lines as follows:

a) Mark (√) the rows that do not have assignment
b) Mark (√) the columns (not already marked) that have zeroes in marked rows.
c) Mark (√) the rows (not already marked) that have assignments in marked columns.
d) Repeat (b) and (c) until no marking is required.
e) Draw lines through all unmarked rows and marked columns. If the number of these lines is equal to the order of the matrix then it is an optimum solution otherwise not.

Step 6:- Determine the smallest cost element not covered by the straight lines. Subtract this smallest cost element from all the uncovered elements and add this to all those element which are lying in the intersection of these straight lines and do not change the remaining elements which lie on the straight lines.

Step 7:- Repeat steps(1) to (6), until an optimum assignment is attained.

Note1:- Incase some rows or columns contain more than one zero, encircle any unmarked zero arbitrarily and cross all other zeroes in its column or row. Proceed in this way until no zero is left unmarked or encircled.

Note 2:- The above assignment algorithm is only for minimization problems.

Note 3:- If the given assignment problem is of maximization type, convert it to a minimization assignment problem by max Z= - min (-Z) and multiply all the given cost elements by -1 in the cost matrix and then solve by assignment algorithm.

Note 4:- Sometimes, a final cost matrix contains more than required number of zeroes at independent positions. This implies that there is more than one optimal solution (multiple optimal solutions) with the same optimum assignment cost.

Example 1:- Consider the problem of assigning five jobs to five persons. The assignment costs are given as follows:

$$\begin{array}{ccc} & & \textbf{Job} \\ & & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \textbf{Person} & \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 8 & 4 & 2 & 6 & 1 \\ 0 & 9 & 5 & 5 & 4 \\ 3 & 8 & 9 & 2 & 6 \\ 4 & 3 & 1 & 0 & 3 \\ 9 & 5 & 8 & 9 & 5 \end{bmatrix} \end{array}$$

Determine the optimum assignment schedule.

Solution:- The cost matrix of the given assignment problem is

$$\begin{bmatrix} 8 & 4 & 2 & 6 & 1 \\ 0 & 9 & 5 & 5 & 4 \\ 3 & 8 & 9 & 2 & 6 \\ 4 & 3 & 1 & 0 & 3 \\ 9 & 5 & 8 & 9 & 5 \end{bmatrix}$$

Since the number of rows is equal to the number of columns in the cost matrix, the given assignment problem is balanced.

Step 1:- Select the smalled cost element in each row and subtract this from all the elements of the corresponding row, we get the reduced matrix.

$$\begin{bmatrix} 7 & 3 & 1 & 5 & 0 \\ 0 & 9 & 5 & 5 & 4 \\ 1 & 6 & 7 & 0 & 4 \\ 4 & 3 & 1 & 0 & 3 \\ 4 & 0 & 3 & 4 & 0 \end{bmatrix}$$

Step 2:- Select the smallest cost element in each column and subtract this from all the elements of the corresponding column, we get the reduced matrix.

$$\begin{bmatrix} 7 & 3 & 0 & 5 & 0 \\ 0 & 9 & 4 & 5 & 4 \\ 1 & 6 & 6 & 0 & 4 \\ 4 & 3 & 0 & 0 & 3 \\ 4 & 0 & 2 & 4 & 0 \end{bmatrix}$$

Since each row and each column contains atleast one zero, we shall make assignments in the reduced matrix.

Step 3:- Examine the rows successively until a row with exactly one unmarked zero is found. Since the 2nd row contains a single zero, encircle this zero and cross all other zeros of its column. The 3rd row contains exactly one unmarked zero, so encircle this zero and cross all other zeros in its column. The 4th row contains exactly one unmarked zero, so encircle this zero and cross all other zeros in its column. The 1st row contains exactly one unmarked zero, so encircle this zero and cross all other zeros in its column. Finally the last row contains exactly on unmarked zero, so encircle this zero and cross all other zeros in its column. Like-wise examine the columns successively. The assignments in rows and columns in th reduced matrix is given by

$$
\begin{bmatrix}
7 & 3 & \cancel{0} & 5 & (0) \\
(0) & 9 & 4 & 5 & 4 \\
1 & 6 & 6 & (0) & 4 \\
4 & 3 & (0) & \cancel{0} & 3 \\
4 & (0) & 2 & 4 & \cancel{0}
\end{bmatrix}
$$

Step 4 :- Since each row and each column contains exactly one assignment(i.e., exactly one encircled zero)the curren assignment is optimal.

Therefore, The optimum assignment schedule is given by A→ 5, B →1, C→4, D→3, E→2.

The optimum (minimum) assignment cost (1+ 0+2+1+5) cost units = 9 units of cost

Example 2: The processing time in hours for the jobs when allocated to the different machines are indicated below. Assign the machines for the jobs so that the total processing time is minimum

		Machines				
		M1	**M2**	**M3**	**M4**	**M5**
	J1	19	9	22	58	11
	J2	63	43	78	72	50
Jobs	**J3**	45	41	28	91	37
	J4	39	74	42	27	49
	J5	25	36	11	57	22

Solution:: The cost matrix of the given problem is

$$\begin{bmatrix} 9 & 22 & 58 & 11 & 19 \\ 43 & 78 & 72 & 50 & 63 \\ 41 & 28 & 91 & 37 & 45 \\ 74 & 42 & 27 & 49 & 39 \\ 36 & 11 & 57 & 22 & 25 \end{bmatrix}$$

Since the number of rows is equal to the number of columns in the cost matrix, the given assignment problem is balanced.

Step 1:- Select the smallest cost element in each row and subtract this from all the elements of the corresponding row, we get the reduced matrix.

$$\begin{bmatrix} 0 & 13 & 49 & 2 & 10 \\ 0 & 35 & 29 & 7 & 20 \\ 13 & 0 & 63 & 9 & 17 \\ 47 & 15 & 0 & 22 & 12 \\ 25 & 0 & 46 & 11 & 14 \end{bmatrix}$$

Step 2:- Select the smallest cost element in each column and subtract this from all the elements of the corresponding column, we get this following reduced matrix

$$\begin{bmatrix} \cancel{0} & 13 & 49 & 0 & 0 \\ 0 & 35 & 29 & 5 & 10 \\ 13 & 0 & 63 & 7 & 7 \\ 47 & 15 & 0 & 20 & 2 \\ 25 & \cancel{0} & 46 & 9 & 4 \end{bmatrix}$$

Step 3:- Now we shall examine the rows successively. Second row contains a single unmarked zero, encircle this zero and cross all other zeros in its column. Third row contains a single unmarked zero, encircle this zero and cross all other zeroes in its column. Fourth row contains a single unmarked zero, encircle this zero and cross all other zeros in its column. After this no row is with exactly one unmarked zero. So go for columns.

Examine the columns successively. Fourth column contains exactly one unmarked zero, encircle this zero and cross all other zeros in its row. After examining all the rows and columns, we get

$$\begin{bmatrix} \cancel{0} & 13 & 49 & (0) & \cancel{0} \\ (0) & 35 & 29 & 5 & 10 \\ 13 & (0) & 63 & 7 & 7 \\ 47 & 15 & (0) & 20 & 2 \\ 25 & \cancel{0} & 46 & 9 & 4 \end{bmatrix}$$

Step 4:- Since the 5^{th} row and 5^{th} column do not have any assignment, the current assignment is not optimal.

Step 5:- Cover all the zeros by drawing a minimum number of straight lines as follows:

a) Mark(√) the rows that do not have assignment. The row 5 is marked.
b) Mark (√) the columns(not already marked) that have zeros in marked rows. Thus column 2 is marked.
c) Mark(√) the rows(not already marked) that have assignments in marked columns. Thus row 3 is marked.
d) Repeat (b) and (c) until no more marking is required. In the present case this repetition is not necessary.
e) Draw lines through all unmarked rows(rows 1, 2 and 4) and marked columns(column 2). We get

$$\begin{bmatrix} 0 & 13 & 49 & 0 & 0 \\ 0 & 35 & 29 & 5 & 10 \\ 13 & 0 & 63 & 7 & 7 \\ 47 & 15 & 0 & 20 & 2 \\ 25 & 0 & 46 & 9 & (4) \end{bmatrix}$$

Step 6:- Here 4 is the smallest element not covered by these straight lines. Subtract this 4 from all the uncovered elements and add this 4 to all those elements which are lying in the intersection of these straight lines and do not change the remaining elements which lie on these straight lines, we get the following matrix.

$$\begin{bmatrix} 0 & 17 & 49 & 0 & 0 \\ 0 & 39 & 29 & 5 & 10 \\ 9 & 0 & 59 & 3 & 3 \\ 47 & 19 & 0 & 20 & 2 \\ 21 & 0 & 42 & 5 & 0 \end{bmatrix}$$

Since each row and each column contains atleast one zero. we examine the rows and columns successively, i.e., repeat step 3 above, we get

$$\begin{bmatrix} \cancel{0} & 17 & 49 & (0) & \cancel{0} \\ (0) & 39 & 29 & 5 & 10 \\ 9 & (0) & 59 & 3 & 3 \\ 47 & 19 & (0) & 20 & 2 \\ 21 & \cancel{0} & 42 & 5 & (0) \end{bmatrix}$$

In the above matrix, each row and each column contains exactly one assignment (i.e., exactly one encircled zero), therefore the current assignment is optimal.

Therefore the optimum assignment schedule is $J_1 \rightarrow M_4$, $J_2 \rightarrow M_1$, $J_3 \rightarrow M_2$, $J_4 \rightarrow M_3$, $J_5 \rightarrow M_5$ and the optimum processing time = 11+43+28+27+25 hours = 134hours

If the number of rows is not equal to the number of columns in the cost matrix of the given assignment problem, then the given assignment problem is said to be unbalanced.

First convert the unbalanced assignment problem into a balanced one by adding dummy rows or dummy columns with zero cost elements in the cost matrix depending upon whether $m < n$ or $m > n$ and then solve by the usual method.

Example 3:- A company has four machines to do three jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table

		Machines			
		1	2	3	4
Job	A	18	24	28	32
	B	8	13	17	19
	C	10	15	19	22

What are job assignments which will minimize the cost?

Solution:- The cost matrix of the given assignment problem is

$$\begin{bmatrix} 18 & 24 & 28 & 32 \\ 8 & 13 & 17 & 19 \\ 10 & 15 & 19 & 22 \end{bmatrix}$$

Since the number of rows is less than the number of columns in the cost matrix, the given assignment problem is unbalanced. To make it balanced, add a dummy job(row) with zero cost elements. The balanced cost matrix is given by

$$\begin{bmatrix} 18 & 24 & 28 & 32 \\ 8 & 13 & 17 & 19 \\ 10 & 15 & 19 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Select the smallest cost element in each row and subtract this from all the elements of the corresponding row, we get the reduced matrix

$$\begin{array}{cccc} 0 & 6 & 10 & 14 \\ 0 & 5 & 9 & 11 \\ 0 & 5 & 9 & 12 \\ 0 & 0 & 0 & 0 \end{array}$$

In this reduced matrix, we shall make the assignments in rows and columns having single zero. We have

$$\begin{array}{cccc} (0) & 6 & 10 & 14 \\ \cancel{0} & 5 & 9 & 11 \\ \cancel{0} & 5 & 9 & 12 \\ \cancel{0} & (0) & \cancel{0} & \cancel{0} \end{array}$$

Since there are some rows and columns without assignment, the current assignment is not optimal. Cover all the zeros by drawing a minimum number of straight lines. Choose the smallest cost element not covered by these straight lines.

$$\begin{array}{ccccc} 0 & 6 & 10 & 14 & \surd \\ 0 & 5 & 9 & 11 & \surd \\ 0 & (5) & 9 & 12 & \surd \\ 0 & 0 & 0 & 0 & \\ \surd & & & & \end{array}$$

Here 5 is the smallest cost element not covered by these straight lines. Subtract this 5 from all the uncovered elements, add this 5 to those elements which lie in the intersection of these straight lines and do not change the remaining elements which lie on the straight lines. We get

$$\begin{array}{cccc} 0 & 1 & 5 & 9 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & 4 & 7 \\ 5 & 0 & 0 & 0 \end{array}$$

Since each row and each column contains atleast one zero we shall make assignment in the rows and columns having single zero. We get

$$
\begin{array}{cccc}
(0) & 1 & 5 & 9 \\
\cancel{0} & (0) & 4 & 6 \\
\cancel{0} & \cancel{0} & 4 & 7 \\
5 & \cancel{0} & (0) & \cancel{0}
\end{array}
$$

Since there are some rows and columns withou assignment, the current assignment is not optimal. Cover al the zeros by drawing a minimum number ofstraight lines.

$$
\begin{array}{ccccc}
0 & 1 & 5 & 9 & \surd \\
0 & 0 & 4 & 6 & \surd \\
0 & 0 & (4) & 7 & \surd \\
5 & 0 & 0 & 0 & \\
\surd & \surd & & &
\end{array}
$$

Choose the smallest cost element not covered by thes straight lines, subtract this from all the uncovered elements add this to those elements--which are in the intersection o the lines and do not change the remaining elements whicl lie on these straight lines. Thus we get-

$$
\begin{array}{cccc}
0 & 1 & 1 & 5 \\
0 & 0 & 0 & 2 \\
0 & 0 & 0 & 3 \\
9 & 4 & 0 & 0
\end{array}
$$

Since each row and each column contains atleast one zerc we shall make the assignment in the rows and column having single zero. We get

$$
\begin{array}{cccc}
(0) & 1 & 1 & 5 \\
\cancel{0} & (0) & \cancel{0} & 2 \\
\cancel{0} & \cancel{0} & (0) & 3 \\
9 & 4 & \cancel{0} & (0)
\end{array}
$$

Since each row and each column contains exactly on assignment the current assignment is optimal. Therefor optimum assignment schedule is given by A→1, B→2 C→3, D→4 and the optimum assignmen cost=(18+13+19+0) cost units

= 50 /- units of cost.

Note1:- For this problem, the alternative optimum schedule is A→1, B→3, C→2, D→4, with the same optimum assignment cost = Rs. (18+17+15+0) = 50 /- units of cost.

Maximization case in Assignment Problems:-

The maximization problem has to be converted into an equivalent minimization problem and then solve by the usual Hungarian method. The conversion of the maximization problem into an equivalent minimization problem can be done by any one of the following methods:

1) Since max Z= -min(-Z), multiply all the cost elements c_{ij} of the cost matrix by -1.
2) Subtract all the cost elements c_{ij} of the cost matrix from the highest cost element in that cost matrix.

Example 4:- A company has a team of four salesmen and there are four districts where the company wants to start its business. After taking into account the capabilities of salesman and the nature of districts, the company estimates that the profit per day in rupees for each salesman in each district is as below:

		Districts			
		1	2	3	4
	A	16	10	14	11
	B	14	11	15	15
Salesmen	**C**	15	15	13	12
	D	13	12	14	15

Find the assignment of salesmen to various districts which will yield maxium profit.

Solution:-The cost matrix of the given assignment problem is

16	10	14	11
14	11	15	15
15	15	13	12
13	12	14	15

Since this is a maximization problem, it can be converted into an equivalent minimization problem by subtracting all the cost elements in the cost matrix from the highest cost 16 of this cost matrix. Thus the cost matrix of the equivalent minimization problem is

0	6	2	5
2	5	1	1
1	1	3	4
3	4	2	1

Select the smallest cost element in each row(column) and subtract this from all the cost elements of the corresponding row(column). We get the reduced cost matrix

0	6	2	5
1	4	0	0
0	0	2	3
2	3	1	0

Since each row and each column contains atleast one zero, we shall make the assignment in rows and columns having single zero. We get

(0)	6	2	5
1	4	(0)	$\cancel{0}$
$\cancel{0}$	(0)	2	3
2	3	1	(0)

Since each row and each column contains exactly one assignment the current assignment is optimal. Therefore optimum assignment schedule is given by A→1, B→3, C→2, D→4 and the optimum profit is =Rs (16 +15 +15 +15)

= Rs. 61/-

Scheduling by PERT and CPM

Introduction

A Project is defined as a combination of interrelated activities all of which must be executed in a certain order to achieve a set goal. A large and complex project involves usually a number of interrelated activities requiring men, machines and materials. It is impossible for the management to make and execute an optimum schedule for such a project just by intuition, based on the organizational capabilities and work experience. A systematic scientific approach has become a necessity for such projects. So a number of methods applying network scheduling techniques has been developed: **Programme Evaluation Review Technique** (PERT) and **Critical Path** Method (CPM) are two of the many network techniques which are widely used for planning, scheduling and controlling large complex projects.

The three main managerial functions for any project are

1. Planning
2. Scheduling
3. Control

Planning

This phase involves a listing of tasks or jobs that must be performed to complete a project under consideration. In this phase, men, machines and materials required for the project in addition to the estimates of costs and durations of various activities of the project are also determined.

Scheduling

This phase involves the laying out of the actual activities of the projects in a **logical sequence** of time in which they have to be performed.

Men and material requirements as well as the **expected completion time** of each activity at each stage of the project are also determined.

Control :

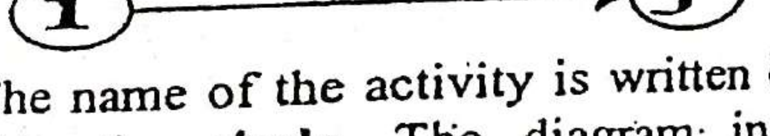
This phase consists of reviewing the progress of the project whether the actual performance is according to the planned schedule and finding the reasons for difference, if any, between the schedule and performance. The basic aspect of control is to analyse and correct this difference by taking remedial action wherever possible.

PERT and CPM are especially useful for scheduling and controlling.

Basic Terminologies

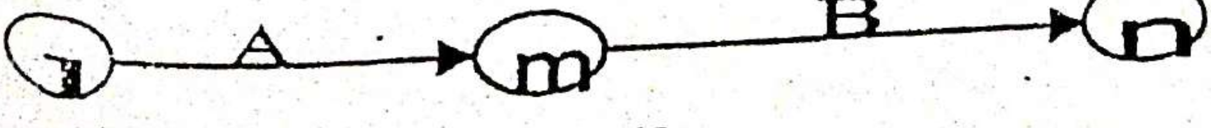
Activity is a task or an item of work, to be done in a project. An activity consumes resources like time, money, labour etc.

An activity is represented by an arrow with a node (event) at the beginning and a node (event) at the end indicating the start and termination (finish) of the activity. Nodes are denoted by circles. Since this is a logical diagram length or shape of the arrow has no meaning. The direction indicates the progress of the activity. Initial node and the terminal node are numbered as $i - j$ ($j > i$) respectively. For example If A is the activity whose initial node is i and the terminal node is j then it is denoted diagrammatically by

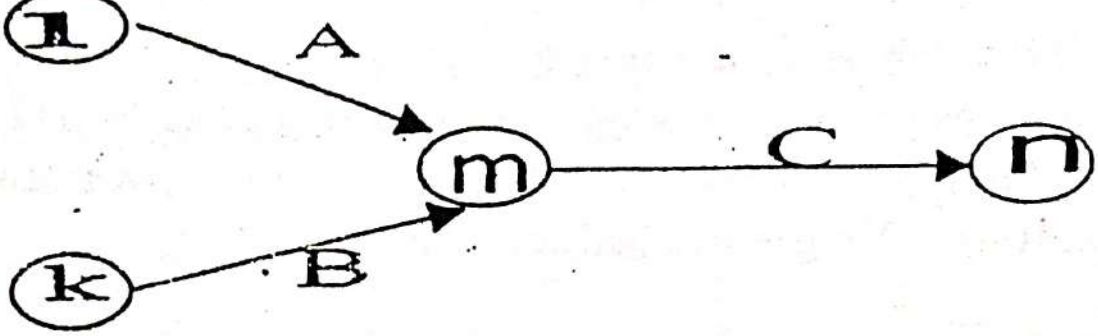


The name of the activity is written over the arrow, **not inside the circle**. The diagram in which arrow represents an activity is called **arrow diagram**. The initial and terminal nodes of activities are also called tail and head events.

If an activity B can start immediately after an activity A then it is denoted by



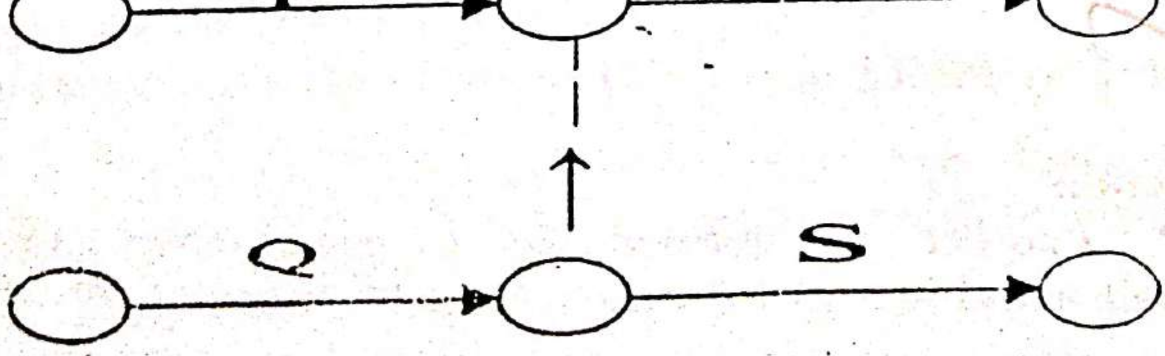
A is called the **immediate predecessor** of B an B is called the **immediate successor** of A. If C can start only after completing activities A and B then it is diagrammatically represented as follows:



Notation: "A is a predecessor of B" is denoted as "A < B". "B is a successor of A" is denoted by "B > A".

If the project contains two or more activities which have some of their immediate predecessors in common then there is a need for introducing what is called **dummy activity**. Dummy activity is an imaginary activity which does not consume any resource and which serves the purpose of indicating the predecessor or successor relationship clearly in any activity on arrow diagram. The need for a dummy activity is illustrated by the following usual example.

Let P, Q be the predeccessors of R and Q be the only predecessor of S.



Activities which have no predeccessors are called **start activities** of the project. All the start activities can be made to have the **same initial node.** Activities which have no **successors** are called **terminal activities** of the project. These can be made to have **the same** terminal node (end node) of the project.

A project consists of a number of activities to be performed in some technological sequence. For example while constructing a building the activity of laying the foundation should be done before the activity of erecting the walls for the building. The diagram denoting all the activities of a project by arrows taking into account the technological sequence of the activities is called the **project network** represented by **activity on arrow diagram** or simply **arrow diagram.**

Note: There is another representation of a project network representing activities on nodes called AON diagram. To avoid confusion we use only activity on arrow diagram throughout the text.

Rules for constructing a project network

1. There must be no loops. For example, the activities F, D, E. Obviously form a loop which is obviously not possible in any real project network.

2. Only one activity should connect any two nodes.

3. No dangling should appear in a project network. i.e., no node of any activity except the terminal node of the project should be left without any activity emanating from it. Such a node can be joined to the terminal node of the project to avoid.

Nodes may be numbered using the rule given below:

(Ford and Fulerson's Rule)

1. Number the start node which has no predecessor activity, as 1.
2. Delete all the activites emanating from this node 1.
3. Number all the resulting start nodes without any predecessor as 2, 3,.....
4. Delete all the activities originating from the start nodes 2, 3, ... in step 3.
5. Number all the resulting new start nodes without any predecessor next to the last number used in step (3).
6. Repeat the process until the terminal node without any successor activity is

   reached and number this terminal node suitably.

Network Computations(earliest completion of a project and critical path)

The completion time of the project is one of the very important things to be calculated knowing the durations of each activity. In real world situation the duration of any activity has an element of uncertainity because of sudden unexpected shortage of labour, machines, materials etc. Hence the completion time of the project also has an element of uncertainity. We first consider the situation where the duration of each activity is deterministic without taking the uncertainity into account.

The first network calculation one does is the computation of earliest start and earliest finish (completion) time of each activity given the duration of each activity The method used is called **forward pass calculation.**

Earliest start of an activity i-j can be denoted as ES or ES_{ij}. It can also be called earliest occurrence of the ever i.

Formula for **Earliest Start of an activity i-j** in a project network is given by

$$ES_j = \text{Max}\,[ES_i + t_{ij}]$$

where ES_i denotes the earliest start time of all activities emanating from node i and t_{ij} is the estimated duration of the activity i-j.

Earliest finish of any activity i-j is got by adding the duration of the activity denoted by t_{ij} to the earliest start of i-j.

To computer the latest finish and latest start of each activity

The method used here is called backward pass calculation since we start with the terminal activity and go back to the very first node. We calculate the latest finish of each activity as follows:

Latest finish of all the terminating(end) activities is taken as the earliest completion time of the project. Similarly latest finish of all the start activities is obviously taken as the same as the earliest start of these start activities. Latest finish of an activity is denoted by LF_j or LF_{ij}. It can also be called the latest occurrence of the event j. *Latest start* of each activity is the latest finish of that activity minus the duration of that activity.

Formula for the *latest start time* of all the activities emanating from the event i of the activity i-j. LS_i = $Min[LS_j - t_{ij}]$ for all defined i-j activities where t_{ij} is the estimated duration of the activity i-j.

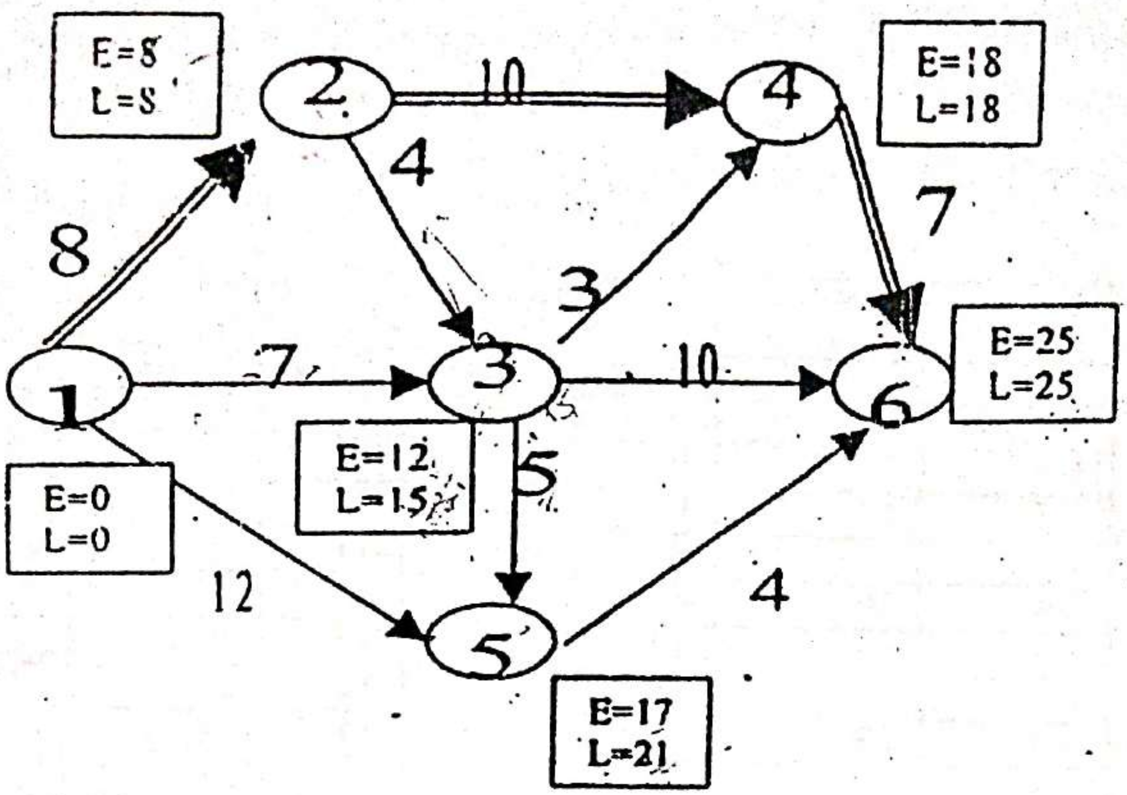
Critical Path : **Paths**, connecting the first initial node to the very last terminal node, of **longest duration** in any project network is called the **Critical path**. This plays a very important role in project scheduling problems.

Example :- Calculate the earliest start, earliest finish, latest start and latest finish of each activity of the project given below and determine the Critical path of the project.

Activity	1-2	1-3	1-5	2-3	2-4	3-4	3-5	3-6	4-6	5-6
Duration (in Weeks)	8	7	12	4	10	3	5	10	7	4

Solution:-

Activity	Duration (in weeks)	Earliest		Latest	
		Start	Finish	Start	Finish
1-2	8	0	8	0	8
1-3	7	0	7	8	15
1-5	12	0	12	9	21
2-3	4	8	12	11	15
2-4	10	8	18	8	18
3-4	3	12	15	15	18
3-5	5	12	17	16	21
3-6	10	12	22	15	25
4-6	7	18	25	18	25
5-6	4	17	21	21	25



Floats:-

Total float of an activity (T.F) is defined as the **difference** between the latest finish and the **earliest finish of the activity** or the difference between the **latest start** and the **earliest start** of the activity.

Total float of an activity $i - j = (LF)_{ij} - (EF)_{ij}$

Or $= (LS)_{ij} - (ES)_{ij}$

Total float of an activity is the amount of time by which that particular activity may be delayed without affecting the duration of the project. If the total float is positive then it may indicate that the resources for the activity are more than adequate. If the total float of an activity is zero it may indicate that the resources are just adequate for that activity. If the total float is negative, it may indicate that the resources for that activity are inadequate.

Note: (L – E) of an event of i – j is called the slack of the event j.

There are three other types of floats for an activity, namely, Free float, Independent float and interference (interfering) float.

Free Float of an activity (F.F.) is that **portion of the total float** which can be used for rescheduling that activity without affecting the succeeding activity. It can be calculated as follows:

Free float of an activity i – j = Total float of i – j – (L – E) of the event j

= Total float of i – j – Slack of the head event j

= Total float of I – j – Slack of the head event j

Where L = Latest occurrence

E = Earliest occurrence

Obviously Free Float ≤ Total float for any activity.

Independent float (I.F) of an activity is the amount of time by which the activity can be rescheduled without affecting the preceding or succeeding activities of that activity.

Independent float of an activity i – j = Free float of i – j – (L–E) of event i.

= Free float of i – j – Slack of the tail event i.

Clearly,

Independent float ≤ Free float for any activity

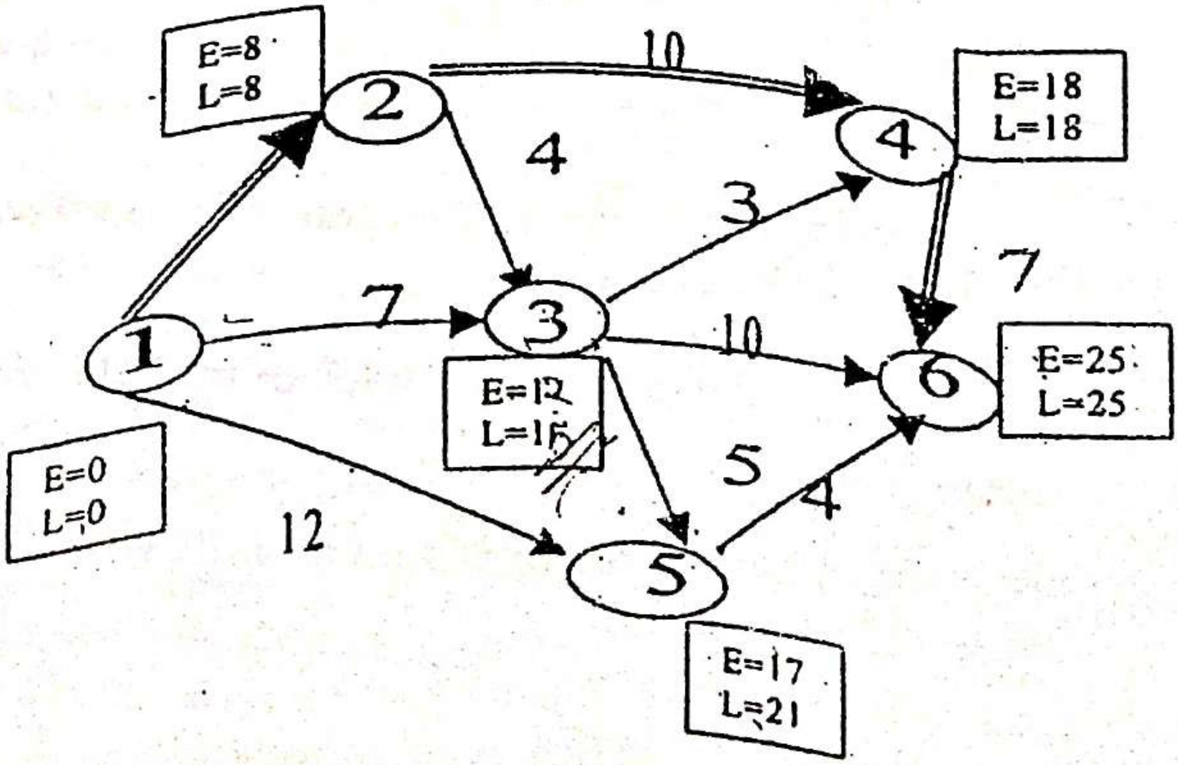
Thus I.F ≤ F.F ≤ T.F.

Interfering Float or Interference Float of an activity i – j is nothing but the slack of the head event j.

Obviously,
Interfering Float i – j = Total Float of i – j – Free Float of i – j.

Example: Calculate the total float, free float and independent float for the project whose activities are given below:

Activity	1-2 3-5	1-3 3-6	1-5 4-6	2-3 5-6	2-4	3-4
Duration (in weeks)	8 5	7 10	12 7	4 4	10	3



Activity	Duration (in weeks)	Earliest		Latest		Floats		
		Start	Finish	Start	Finish	TF	FF	IF
1-2	8	0	8	0	8	0	0	0
1-3	7	0	7	8	15	3	5	5
1-5	12	0	12	9	21	9	5	5
2-3	4	8	12	11	15	3	0	0
2-4	10	8	18	8	18	0	0	0
3-4	3	12	15	15	18	3	3	0
3-5	5	12	17	16	21	4	0	3
3-6	10	12	22	15	25	3	3	0
4-6	7	18	25	18	25	0	0	0
5-6	4	17	21	21	25	4	4	0

Total float of (2-3) = (LF-EF) of (2-3) = 15-12 =3 from the table against the activity 2-3

Free float of (2-3) = Total float of (2-3) – (L-E) of event 3

= 3 – (15 - 12) from the figure for event 3

= 0

Independent float of (1-5) = Free Float of (1-5) – (L-E) of event 1

= 5 – (0 – 0) = 5

Note that all the critical activities have their total float as zero. The critical path can also be defined as the path of least (zero) total float.

Uses of floats:- Floats are useful in resource leveling and resource allocation problems. Floats give some flexibility in rescheduling some activities so as to smoothen the level of resources or allocate the limited resources as best as possible.

Exercise:-

1. Solve the assignment problem

	A	B	C	D
I	1	4	6	3
II	9	7	10	9
III	4	5	11	7
IV	8	7	8	5

2. Write down the stepwise procedure for determing the critical path of a project.

3. A project schedule has the following characteristics:

Activity:	1-2	1-3	2-4	3-4	3-5	4-9	5-6	5-7	6-8	7-8
Time:	4	1	1	1	6	5	4	8	1	2

Activity:	8-10	9-10
Time:	5	7

Find the critical path.

4. A maintenance activity consists of the following jobs. Draw the network for the project and calculate the total float and free float for each activity. What can you say about the slacks of the events of the project?

Jobs:	1-2	2-3	3-4	3-7	4-5	4-7	5-6	6-7
Duration:	3	4	4	4	2	2	3	2

5. Draw the network and determine the critical path for the given data:

Jobs:	1-2	1-3	2-4	3-4	3-5	4-5	4-6	5-6
Duration:	6	5	10	3	4	6	2	9

Find the total float, free float and independent float of each activity.

630
1200
450

LESSON 4

Programme Evaluation Review Technique : (PERT)

This technique, unlike CPM, takes into account the uncertainty of project durations into account. PERT calculations depend upon the following three time estimates.

Optimistic (least) time estimate : (t_0 or a) is the duration of any activity when everything goes on very well during the project, i.e., labourers are available and come in time, machines are working properly, money is available whenever needed, there is no scarcity of raw material needed etc.

Pessimistic (greatest) time estimate : (t_p or b) is the duration of any activity when almost every thing goes against our will and a lot of difficulties is faced while doing a project.

Mostly likely time estimate : (t_m or b) is the duration of any activity when sometimes things go on very well, sometimes things go on very bad while doing the project.

Two main assumptions made in PERT calculations are

(i)The activity durations are independent. i.e., the time required to complete an activity will have no bearing on the completion times of times of any other activity of the project.

(ii)The activity durations follow β – distribution. β distribution is a probability distribution with density function with mean $t_e = (1/3)[2t_m + (1/2)(t_0 - t_p)]$ and the standard deviation $\sigma_t = (t_p - t_0)/6$

PERT Procedure

(1) Draw the project net work

(2) Compute the expected duration of each activity
$t_e = (t_0 + 4t_m + t_p)/6$

(3) Compute the expected variance
$\sigma^2 = [(t_p - t_0)/6]^2$ of each activity.

(4) Compute the earliest start, earliest finish, latest start, latest finish and
Total float of each activity.

(5) Determine the critical path and identify critical activities.

(6) Compute the expected variance of the project length (also called the variance of the critical path) σ_c^2 which is the sum of the variances of all the critical activities.

(7) Compute the expected standard deviation of the project length σ_c and calculate the standard normal deviate $(T_S - T_E)/\sigma_c$ where

T_S = Specified or Scheduled time to complete the project
T_E = Normal expected project duration
σ_c = Expected standard deviation of the project length.

(8) Using (7) one can estimate the probability of completing the project within a specified time, using the normal curve (Area) tables.

Basic differences between PERT and CPM

PERT

1. PERT was developed in a brand new R and D project it had no consider and deal with the uncertainities associated with such projects. Thus the project duration is regarded as a random variable and therefore probabilities are calculated so as to characterise it.

2. Emphasis is given to important stages of completion of task rather than the activities required to be performed to reach a particular event or task in the analysis of network. i.e., PERT network is essentially an event – oriented network.

3. PERT is usually used for projects in which time estimates are uncertain. Example : R & D activities which are usually non-repetitive.

4. PERT helps in identifying critical areas in a project so that suitable necessary adjustments may be made to meet the scheduled completion date of the project.

CPM

1. CPM was developed for conventional projects like construction projects which consists of well known routine tasks whose resource

equirement and duration were known with certainty.

2. CPM is suited to establish a trade off for optimum balancing between schedule time and cost of the project.

3. CPM is used for projects involving well known activities of repetitive in nature.

Example :

Three time estimates (in months) of all activities of a project are as given below:

Time in Months

Activity	a	m	b
1-2	0.8	1.0	1.2
2-3	3.7	5.6	9.9
2-4	6.2	6.6	15.4
3-4	2.1	2.7	6.1
4-5	0.8	3.4	3.6
5-6	0.9	1.0	1.1

(a) Find the expected duration and standard deviation of each activity

(b)Construct the project network

(c)Determine the critical path, expected project length and expected variance of the

project length.

(d) What is the probability that the project will be completed

(i) two months later than expected

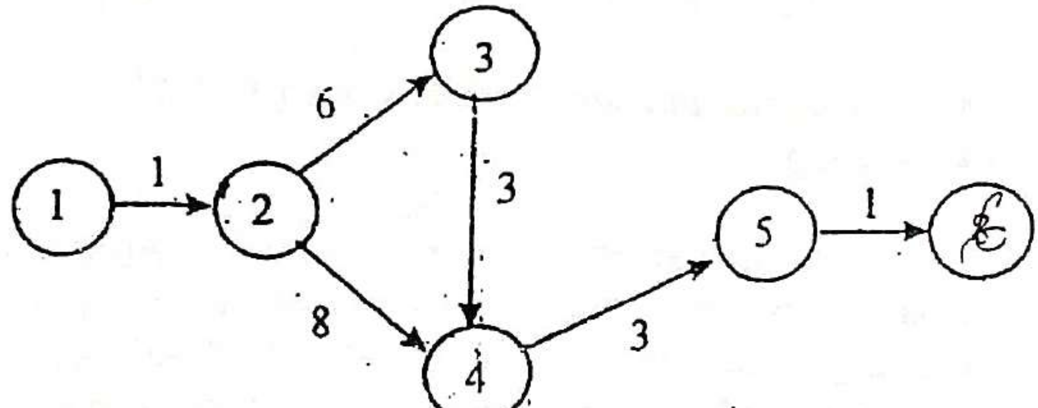
(ii) not more than 3 months earlier than expected

(iii) what due date has about 90% chance of being met?

Solution : (a)

Activity	a	m	b	t_e	σ
1-2	0.8	1.0	1.2	1	0.067
2-3	3.7	5.6	9.9	6	1.03
2-4	6.2	6.6	15.4	8	1.53
3-4	2.1	2.7	5.1	3	0.5
4-5	0.8	3.4	3.6	3	0.47
5-6	0.9	1.0	1.1	1	0.033

b)



c) Critical path: 1-2-3-4-5-6

Expected project length $= 14$ months

Expected Variance $= (0.067)^2 + (1.03)^2 + (0.5)^2 + (0.47)^2 + (0.033)^2$

$= 1.5374$

Therefore $\sigma_c = \sqrt{1.5374} = 1.2399$

d) i) $T_S = 16$, $T_E = 14$, $\sigma_c = 1.2399$

$z = (16-14) / \sigma_c = 2 / \cancel{1.6023}\ 1.2399$

$= 1.6130$

$P(T_S \leq 13) = 0.9463$

$P(T_S \leq 13) = 94.63\%$ = Required probability

d) ii) $T_S = 11$, $T_E = 14$, $\sigma_c = 1.2399$

$$z = (T_S - T_E) / \sigma_c$$
$$= (11-14) / 1.2399 = -2.42$$
$$P(T_S \leq 11) = 0.5 - 0.4922 = 0.0078$$

Required Probability = 0.78%

d)iii) z value for 90% area in the table = 1.28 nearly

Let T_S be the required due date. Then

$$z = (T_S - T_E) / \sigma_c$$
$$1.28 = (T_S - 14) / \cancel{1.6023}\ 1.2399$$
$$T_S = 14 \div 1.28 \times 1.2399 = 15.59 \text{ months nearly.}$$

Cost Consideration in PERT and CPM

Crashing

In all the earlier discussions on PERT and CPM it was assumed that the only constraint for an activity was the starting date and the completion date. Availability of resources and cost aspects were not considered.

There are two kinds of cost with can influence the project schedules. They are **Direct costs** and **Indirect costs**.

Direct costs are the costs directly associated with each activity such as machine costs, labour costs etc for each activity.

Indirect costs are the costs due to management services, rentals, insurance including allocation of fixed expenses, cost of security etc.

Direct costs, increase when the duration of any activity is to be reduced since one has to use more machines, more labour, more money to shorten the duration. Indirect costs decrease when we shorten the duration of a project.

Therefore there is some optimum project duration - a balance between excessive direct costs for shortening the project and excessive indirect costs for lengthening the project.

There is a method for finding the optimum duration and the associated least cost called **least cost schedule**. It should be observed that there is no need to crash all jobs to get a project done faster. Only critical activities need be expedited. Least cost schedule will indicate which **critical activities are to be crashed** and by how much so as to get the optimum duration.

Step by Step procedure for least cost schedule

(1) Draw the network.

(2) Determine the critical path and the normal duration. Also identify the critical activities.

(4) Find the total Normal cost and the Normal duration of the project

(5) Compute the cost slope for each activity by using the formula

$$\text{Cost Slope} = \frac{\text{Crash cost} - \text{Normal cost}}{\text{Normal duration} - \text{Crash duration}}$$

(6) Crash the critical activity of least cost slope first to the maximum extent

Possible so that the project duration is really reduced.

(7) Calculate the new direct d cost by cumulatively adding the cost of the crashing to the current direct cost.

Total Cost = New direct cost + Current Indirect cost.

(8) When critical activities are crashed and the duration is reduced other paths may also become critical. Such critical paths are called **parallel critical paths.** When there is more than one critical path in a network, project duration can be reduced only when either the duration of a critical activity common to all critical paths is reduced or the durations of different suitable activities on different critical paths are simulataneously reduced.

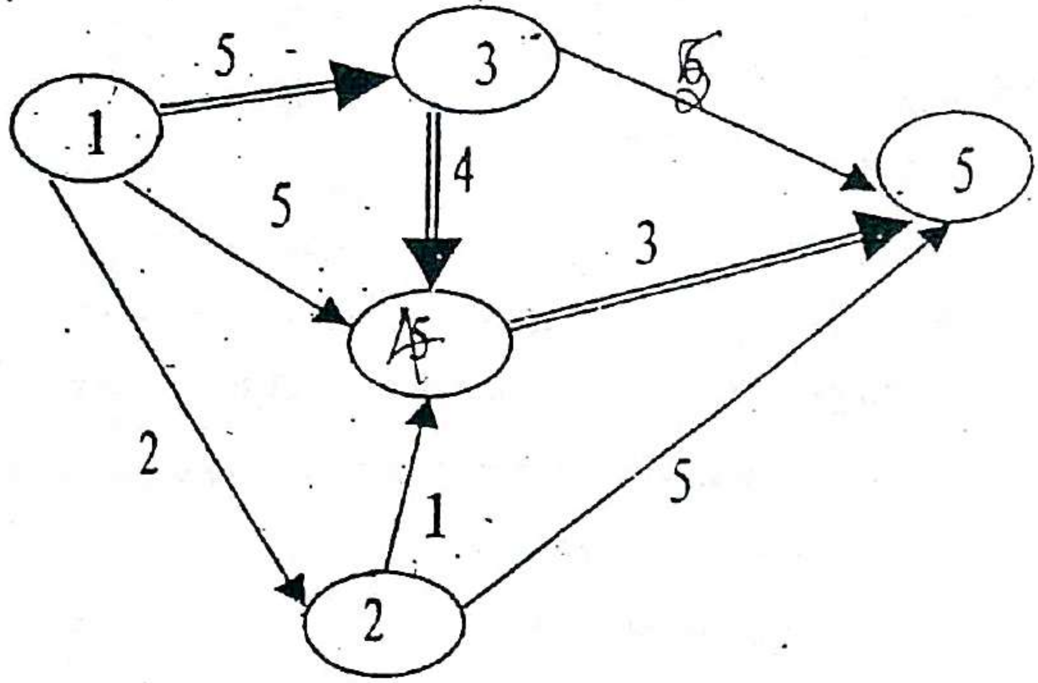
(9) Stop when the total cost is minimum. This gives optimum (least cost) schedule called **optimum duration.**

Note: (1) Minimum (least duration) need not be the same as (least cost) optimum duration. (2) When no indirect cost is given the least cost optimum duration is the same as the normal duration where no crashing is done least crashing cost duration may be found out.

Example : The following time-cost table (time in days, cost in rupees) applies to a project. Use it to arrive at the network associated with completing the project in minimum time at minimum cost.

Activity	Normal		Crash	
	Time	Cost	Time	Cost
1-2	2	800	1	1400
1-3	5	1000	2	2000

1-4	5	1000	3	1800
2-4	1	500	1	500
2-5	5	1500	3	2100
3-4	4	2000	3	3000
3-5	6	1200	4	1400
4-5	3	900	2	1600



Critical path 1-3-4-5

Normal duration = 12 days

Total Normal cost = Rs. 8900

This cost is the minimum cost since no indirect cost is given in the problem. If we want the minimum duration with respect to crashing cost we proceed as follows:
Critical activities are 1-3, 3-4 and 4-5

Cost slope table

Activity	Cost slope
1-2	600
1-3	333.33

$\frac{1400 - 800}{2 - 1}$

$\frac{2000 - 1000}{5 - 2}$

1-4	400	(Cr = Critical
2-4	-	cannot be crashed
2-5	300	
3-4	1000	(Cr)
3-5	200	
4-5	700	(Cr)

Stage 1: Crash the activity 1-3 by 3days.

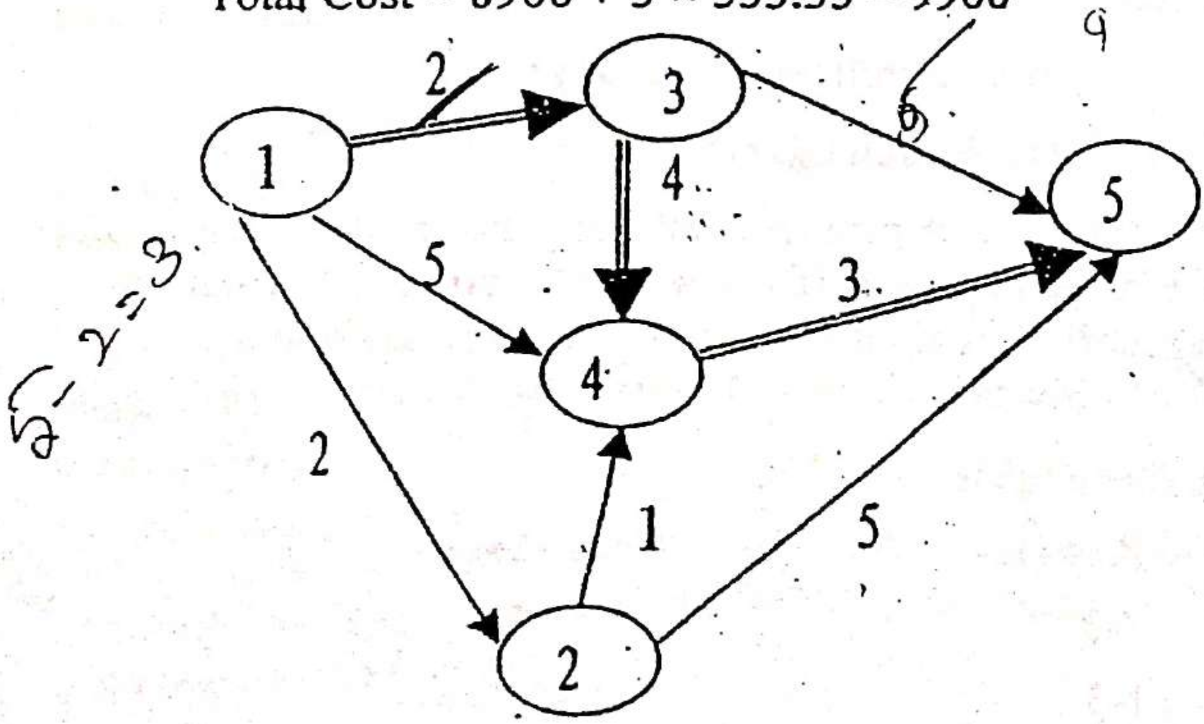
Since 1-3 is the critical activity of least cost slope.

Critical path continues to be the same.

No new critical path.

Now the duration = 9 days

Total Cost = 8900 + 3 × 333.33 = 9900



Stage 2: Crash 4-5 by 1 day

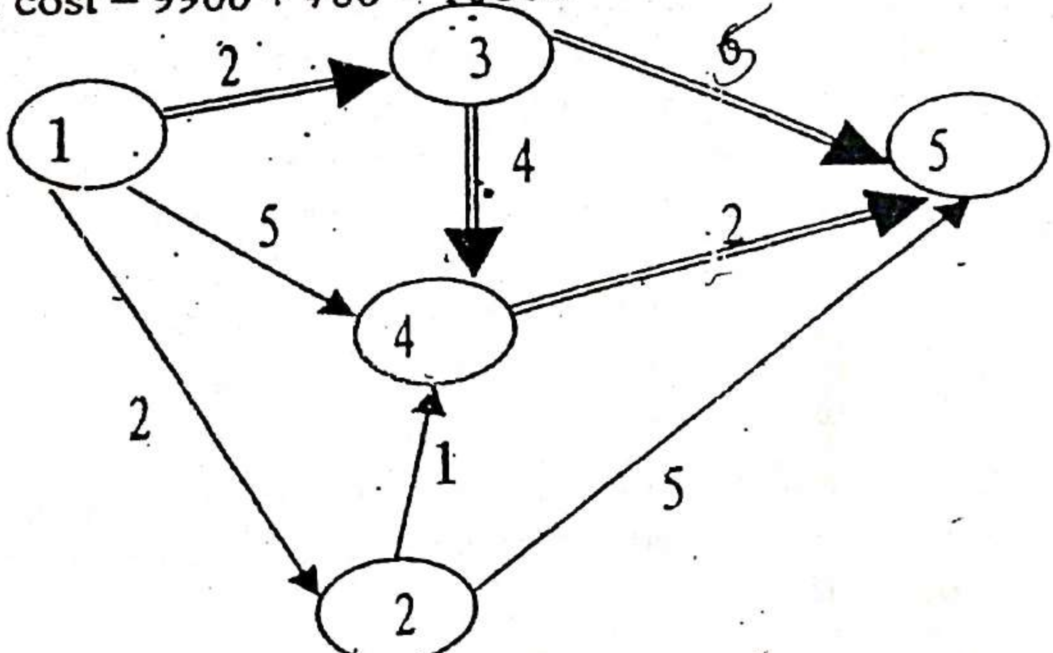
Since 4-5 is the critical activity of least cost slope.

Duration = 8 days

There are 2 critical paths now

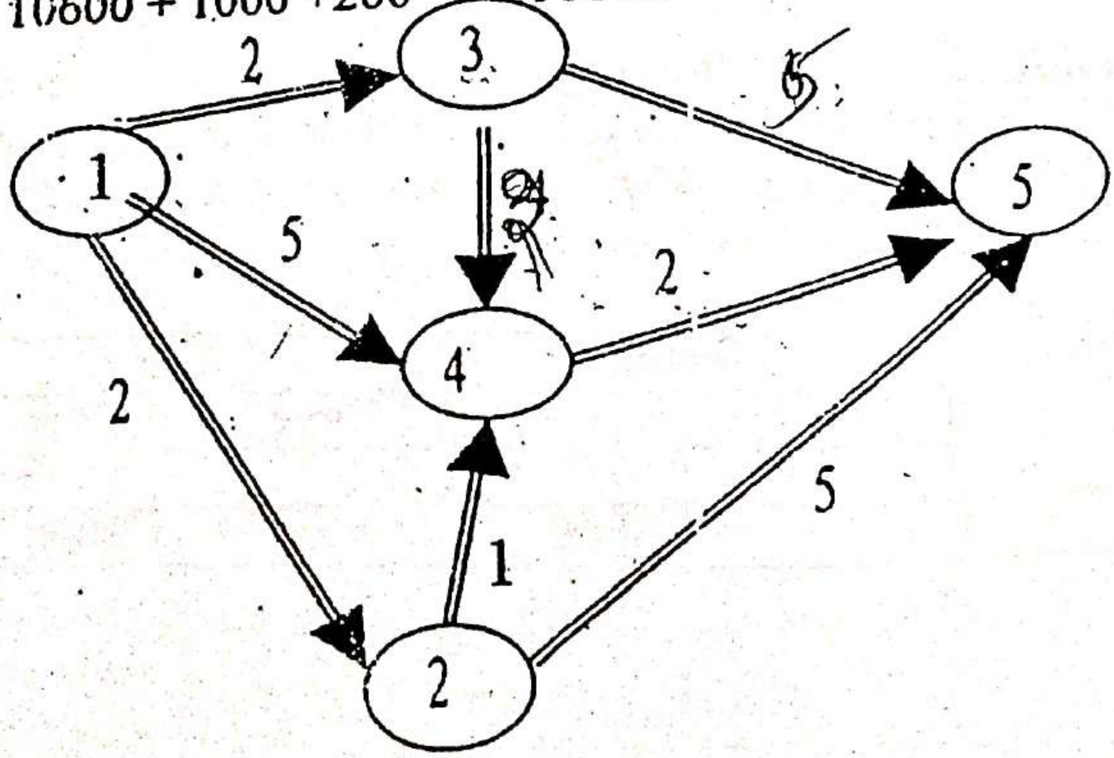
i) 1-3-4-5
ii) 1-3-5

Total cost = 9900 + 700 = 10600



Stage 3:- Crash 3-4 by 1 day and 3-5 by 1 day
Duration = 7 days

Critical paths
i) 1-3-4-5
~~ii) 1-3-5~~
iii) 1-4-5
(iv)1-2-5
Cost = 10600 + 1000 +200 = 11800



crashing is not possible since the activities 1-3, 3-4 and 4-5 can never be crashed further and so the length of the critical path cannot be reduced below seven even though the durations of some other critical activities can be crashed.

Therefore least (Minimum) duration is 7 days with associated cost as Rs. 11800

Least cost crash schedule table

Stage	Crash	Current duration	Direct Cost	Indirect Cost	Total Cost
(0)	0	12	8900	-	8900
(1)	1-3 by 3 days	9	8900 + 3×333.33	-	9900
(2)	4-5 by days	8	9900 + 1×700	-	10600
(3)	3-4 by 1 day and 3-5 by 1 day	7	10600 + 1 × 1000 + 1 × 200	-	11800

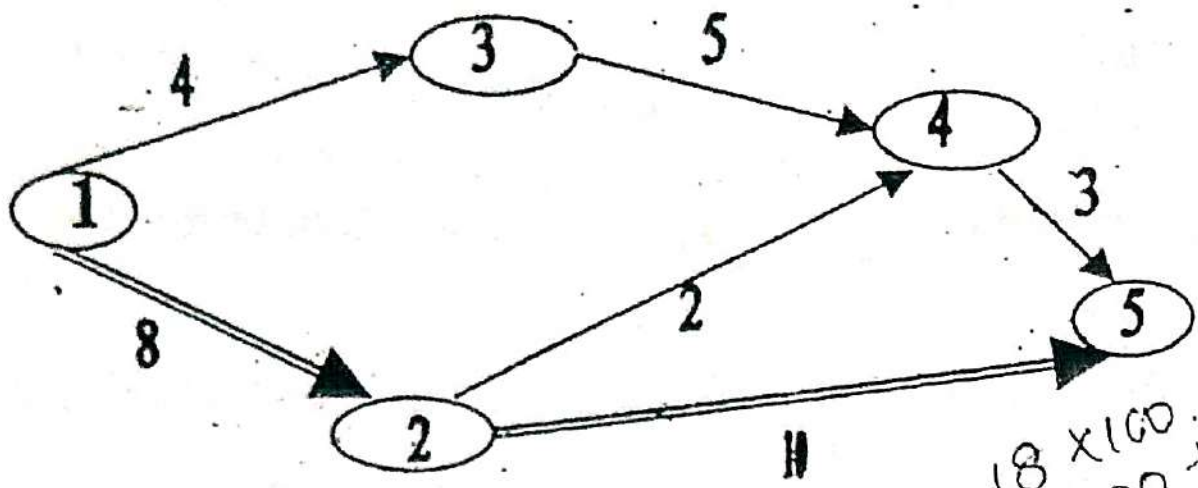
Example : The following data is pertaining to a project with normal time and crash time.

Jobs	Normal		Crash	
	Time	Cost	Time	Cost
1-2	8	100	6	200
1-3	4	150	2	350
2-4	2	50	1	90
2-5	10	100	5	400
3-4	5	100	1	200
4-5	3	80	2	100

90 180 300 130 170

a) If the indirect cost is Rs.100 per day find the least cost schedule(optimum duration)

b) What is the minimum duration?



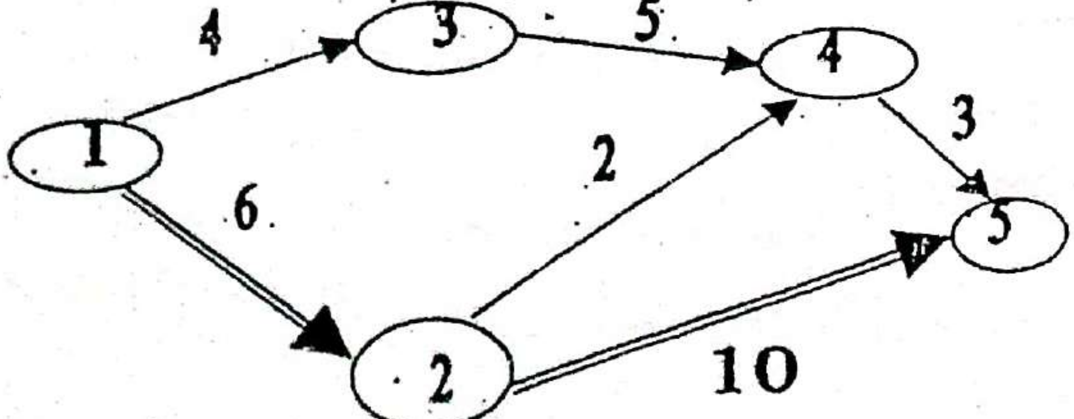
18 x100
1800 +580

Critical path 1-2-5: Normal duration 18 days.
Total cost = Indirect cost +direct cost = 1800 + 580 = Rs. 2380

Cost slope table

Activity	Cost slope	
1-2	50	(Cr = Critical)
1-3	100	
2-4	40	
2-5	75 ~~60~~	(Cr)
3-4	25	
4-5	40	

Stage1:- 1-2 is the critical activity of least cost slope
Crash 1-2 by 2 days. Current Critical path: 1-2-5
Current duration = 18 -2 = 16 days +680
Current Total cost = 16 x 100 + 100 = 1600 +680 = 2280



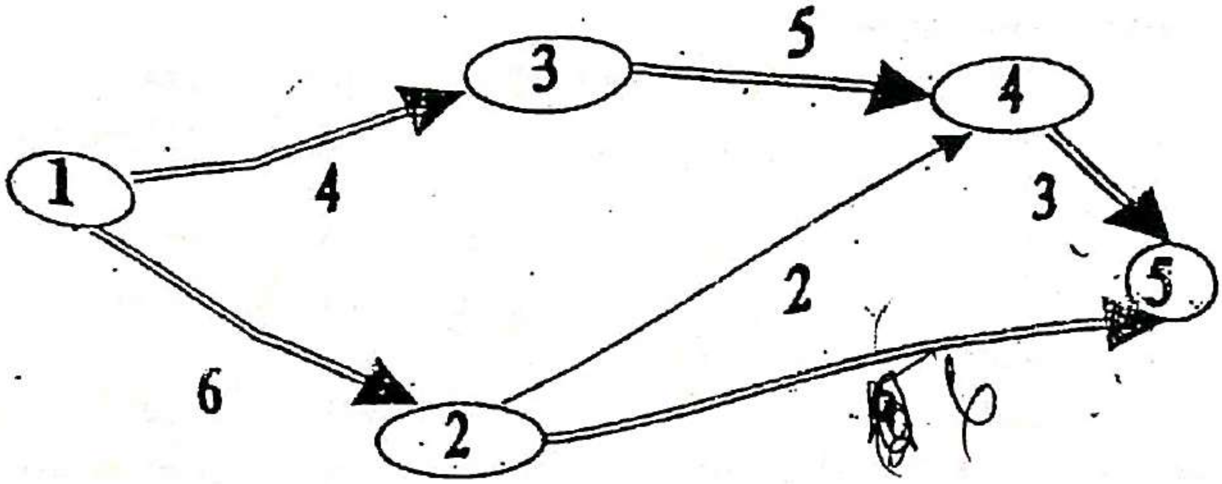
Stage 2:- Critical activities 1-2 and 2-5. Crash 2-5 by 4 days.

Since the duration of the path 1-3-4-5 is 12 days.

Current Critical paths : i) 1-2-5 ii) 1-3-4-5

Current duration = 16 – 4 = 12 days

Current total cost = Rs. 12 x 100 + Rs. 680 + Rs. 240 = Rs. 2120



Stage 3:- Critical activities 1-2, 1-3, 2-5, 3-4 and 4-5.

Crash the critical activities 2-5 and 4-5 by 1 day each since the duration of the path 1-2-4-5 is 11 days and also the activity 2-5 can be crashed only by one day.

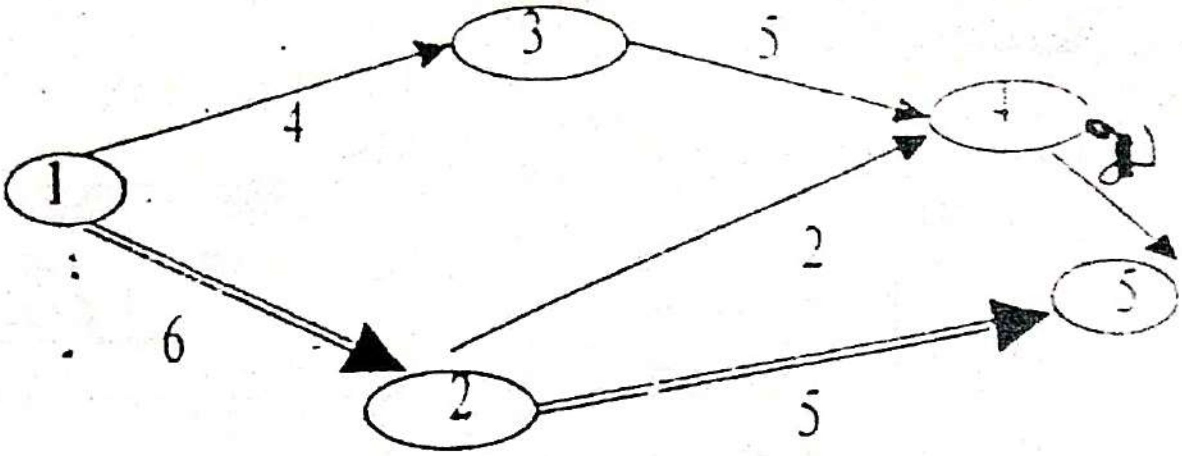
Current Critical paths: i) 1-2-5 and ii) 1-3-4-5

Current duration : 12-1 = 11 days

Current total cost = Rs.11 x 100 + Rs. 920 + 1 x60 + 1 x10 = Rs.2090

No further crashing is possible since all the activities on the critical path 1-2-5 have been crashed to the maximum extent.

Hence the optimum duration = 11 days Least cost = Rs 2090



b)Least or minimum duration is also 11 days.
Crashing schedule can be tabulated as follows:

Stage	Crash	Current Duration	Direct Cost	Indirect Cost	Total Cost
(0)	0	18	580	1800	2380
(1)	1-2 by 2 days	16	680	1600	2280
(2)	2-5 by 4 days	12	920	1200	2120
(3)	2-5 and 4-5 by 1 days	11	990	1100	2090

Exercise

1) What are the two main costs for a project? Illustrate with examples.

2) Define (a) Indirect cost (b) Direct cost for a project

3) The activities of a project with normal and crash time and cost are given below:

Activity	Normal Time hrs	Normal Cost Rs.	Crash Time hrs	Crash Cost Rs.
(1, 2)	20	2000	15	3000
(1, 3)	10	1500	7	2400
(2, 5)	15	1000	10	1500
(3, 4)	16	3000	12	4000
(3, 5)	22	4500	16	5700
(4, 5)	14	1500	10	2100

13500

Find the optimum scheduling of the project which minimizes the total cost.

4) Write Short notes on the cost aspects of PERT.

5) Time estimates for a project is given below:

Activity	Optimistic time	Most-likely time	Pessimistic time (days)
A (1, 2)	2	5	14
B (1, 3)	9	12	15
C (2, 4)	5	14	17
D (3, 4)	2	5	8
E (4, 5)	6	9	12
F (3, 5)	8	17	20

Find the critical path expected project completion time and the probability that the project will be completed in 30 days.

6) A maintenance foreman has given the following estimate of times and cost of jobs in a maintenance project.:

		Normal		Crash	
Job	Predeccessor	Time hrs	Cost Rs.	Time hrs	Cost Rs.
A	-	8	80	6	100
B	A	7	40	4	94
C	A	12	100	5	184
D	A	9	70	5	102
E	B,C,D	6	50	6	50

Overhead cost is Rs.25/- per hour.
Find

(a) The normal duration of the project and the associated cost

(b) The minimum duration (optimum) of the project and associated cost.

(c) The least duration of the project and its cost

(d) If all the activities are crashed what will be the project duration and the corresponding cost