x	x_0	x_1	x_2	...	x_n
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	...	$f(x_n)$

If the value of $f(y)$ is to be found at some point y in the interval $[x_0, x_n]$ and y is not one of the tabulated points, then the value of $f(y)$ is estimated by using the known values of $f(x)$ at the surrounding points. This process of computing the value of a function inside the given range is called **interpolation**. Simply interpolation means insertion or filling up intermediate terms of a series. If the point y lies outside the domain $[x_0, x_n]$ then the estimation of $f(y)$ is called **extrapolation.** In this chapter we will be mainly concerned with interpolation.

❑ Newton's Forward Interpolation Formula

We know that

$$\Delta y_0 = y_1 - y_0 \quad i.e., y_1 = y_0 + \Delta y_0 = (1+\Delta)\, y_0$$

$$\Delta y_1 = y_2 - y_1 \quad i.e., y_2 = y_1 + \Delta y_1 = (1+\Delta)\, y_1 = (1+\Delta)^2 y_0$$

$$\Delta y_2 = y_3 - y_2 \quad i.e., y_3 = y_2 + \Delta y_2 = (1+\Delta)\, y_2 = (1+\Delta)^3 y_0$$

In general, $y_n = (1+\Delta)^n y_0$

Expanding $(1+\Delta)^n$ by using Binomial theorem we have

$$y_n = \{1 + n\Delta + \frac{n(n-1)\Delta^2}{2!} + \frac{n(n-1)(n-2)\Delta^3}{3!} + \ldots\}\, y_0$$

$$y_n = f(x_0 + nh) = y_0 + n\,\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \ldots$$

This result is known as Gregory–Newton forward interpolation (or) Newton's formula for equal intervals.

Note 1 : This formula is very useful and gives greater accuracy when $x_0 + nh$ is near the beginning of the table.

Note 2 : The first two terms of the above formula will give the linear interpolation, the third term will give the parabolic interpolation and so on.

Example 1 : *Find the missing y_x values from the first difference provided.*

y_x	0	–	–	–	–	–
Δy_x	0	1	2	4	7	11

Example 3 : *The following table gives the population of a town during the last six census. Estimate, using Newton's interpolation formula, the increase in the population during the period 1946 to 1948.*

Year	1911	1921	1931	1941	1951	1961
Population (in thousands)	12	13	20	27	39	52

Solution :

The difference table is as follows :

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
$1911(x_0)$	$12\ (y_0)$					
		Δy_0 1				
			$(\Delta^2 y_0)$ 6			
1921	13			$(\Delta^3 y_0)$ -6		
		7			$(\Delta^4 y_0)$ 11	
			0			$(\Delta^5 y_0)$ -20
1931	20			5		
		7			-9	
			5			
1941	27			-4		
		12				
			1			
1951	39					
		13				
1961	52					

Here $x_0 = 1911,\ h = 10,\ y_0 = 12$

By Newton's formula we have

$$y(x_0 + nh) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \ldots$$

$$\begin{aligned}
y(1946) &= ? \\
x_0 + nh &= 1946 \\
1911 + n \cdot 10 &= 1946 \quad i.e.,\ n = 3.5
\end{aligned}$$

$$\begin{aligned}
\therefore y(1946) &= 12 + (3.5)(1) + \frac{(3.5)(3.5-1)}{2} \times 6 \\
&+ \frac{(3.5)(3.5-1)(3.5-2)}{6} \times (-6) \\
&+ \frac{3.5(3.5-1)(3.5-2)(3.5-3)}{24} \times 11 \\
&+ \frac{(3.5)(3.5-1)(3.5-2)(3.5-3)(3.5-4)}{120} \times (-20) \\
&= 12 + 3.5 + 26.25 - 13.125 + 3.0078 + 0.5469 \\
&= 12 + 3.5 + 26.25 + 3.0078 + 0.5469 - 13.125 \\
&= 32.18
\end{aligned}$$

$\therefore$ **The population in the year 1946 is 32.18.**

To find the population in the year 1948 :

By definition...

$$a - 0 = 1 \quad \text{i.e.,} \quad a = 1$$
$$b - a = 2 \Rightarrow b = 2 + a = 3$$
$$c - b = 4 \Rightarrow c = 4 + b = 7$$
$$d - c = 7 \Rightarrow d = 7 + c = 14$$
$$e - d = 11 \Rightarrow e = 11 + d = 25$$

Example 2: ***A function f(x) is given by the following table. Find f (0.2) by a suitable formula.***

x	0	1	2	3	4	5	6
$f(x)$	176	185	194	203	212	220	229

Solution:

The difference table is as follows :

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
$0\ (x_0)$	$176\ (y_0)$	Δy_0					
		9	$(\Delta^2 y_0)$				
1	185		0	$(\Delta^3 y_0)$			
		9		0	$(\Delta^4 y_0)$		
2	194		0		0	$(\Delta^5 y_0)$	
		9		0		−1	$(\Delta^6 y_0)$
3	203		0		−1		5
		9		−1		4	
4	212		−1		3		
		8		2			
5	220		1				
		9					
6	229						

Here $x_0 = 0, h = 1, y_0 = 176 = f(x_0)$

We have to find the value of f (0.2). By Newton's forward interpolation formula we have,

$$y(x_0 + nh) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \cdots$$

$$y(0.2) = ?$$
$$x_0 + nh = 0.2$$
$$0 + n \cdot 1 = 0.2$$
$$n = 0.2$$

$$\therefore y(0.2) = 176 + (0.2)(9) + \frac{(0.2)(0.2-1)}{2} \cdot 0$$
$$= 176 + 1.8$$
$$= 177.8$$

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To find $y(1948)$.

$$x_0 + nh = 1948$$
$$1911 + n \cdot 10 = 1948$$
$$n = 3.7$$

$$y(1948) = 12 + 3.7 + \frac{(3.7)(3.7-1)}{2} \times 6 + \frac{3.7(3.7-1)(3.7-2)}{6} \times (-6) + \frac{3.7(3.7-1)(3.7-2)(3.7-3)}{24} \times 11 + \frac{3.7(3.7-1)(3.7-2)(3.7-3)(3.7-4)}{120} \times (-20)$$

$$= 12 + 3.7 + 29.97 - 16.983 + 5.4487 + 0.5944$$
$$= 34.73$$

The population in the year 1948 is 34.73.

Increase in the population during the period 1946 to 1948.

= Population in 1948 – Population in 1946

= 34.73 – 32.18

= **2.55 thousands**

Example 4 : ***Find the value of $e^{1.85}$, given***

$e^{1.7} = 5.4739$, $e^{1.8} = 6.0496$, $e^{1.9} = 6.6859$,

$e^{2.0} = 7.3891$, $e^{2.1} = 8.1662$, $e^{2.2} = 9.0250$, $e^{2.3} = 9.9742$

Solution :

The difference table is as follows :

x	$e^x = y$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1.7(x_0)	5.4739(y_0)						
		(Δy_0) 0.5757					
1.8	6.0496		($\Delta^2 y_0$) 0.0606				
		0.6363		($\Delta^3 y_0$) 0.0063			
1.9	6.6859		0.0669		($\Delta^4 y_0$) 0.0007		
		0.7032		0.0070		($\Delta^5 y_0$) 0.0001	
2.0	7.3891		0.0739		0.0008		($\Delta^6 y_0$) 0
		0.7771		0.0078		0.0001	
2.1	8.1662		0.0817		0.0009		
		0.8588		0.0087			
2.2	9.0250		0.0904				
		0.9492					
2.3	9.9742						

Example 1 : ***The following table gives the values of a function at equal intervals.***

x	0.0	0.5	1.0	1.5	2.0
$f(x)$	0.3989	0.3521	0.2420	0.1295	0.0540

Evaluate f (1.8).

Solution :

The difference table is as given below.

x	$y = f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
0.0	0.3989				
		–0.0468			
0.5	0.3521		–0.0633		
		–0.1101		0.0609	
1.0	0.2420		–0.0024		–0.0215
		–0.1125		0.0394	$(\nabla^4 y_0)$
1.5	0.1295		0.0370	$(\nabla^3 y_0)$	
		–0.0755	$(\nabla^2 y_0)$		
2.0	0.0540	(∇y_0)			
(x_0)	(y_0)				

Here $x_0 = 2.0$, $y_0 = 0.0540$, $h = 0.5$.

As we have to find the value of $f(1.8)$ near the lower end, we apply Newton's Backward interpolation formula,

$$y(x_0 + nh) = y_0 + n\nabla y_0 + \frac{n(n+1)}{2!}\nabla^2 y_0 + \frac{n(n+1)(n+2)}{3!}\nabla^3 y_0 + \ldots$$

$$y(1.8) = ?$$

$$x_0 + nh = 1.8 \quad \Rightarrow \quad 2.0 + n(0.5) = 1.8$$

$$n = \frac{1.8 - 2.0}{0.5} = -0.4$$

$$\therefore y(1.8) = 0.0540 + (-0.4)(-0.0755) + \frac{(-0.4)(-0.4+1)}{2!}(0.0370)$$

$$+ \frac{(-0.4)(-0.4+1)(-0.4+2)}{3!}(0.0394) + \text{negligible terms}$$

$$= 0.0540 + 0.0302 - 0.0044 - 0.00252 = 0.07728$$

Hence $\quad f(1.8) = \mathbf{0.07728}$

$$y(x) = \frac{(x-1)(x-2)(x-4)(x-5)(x-6)}{(0-1)(0-2)(0-4)(0-5)(0-6)}(1)$$
$$+ \frac{(x-0)(x-2)(x-4)(x-5)(x-6)}{(1-0)(1-2)(1-4)(1-5)(1-6)}(14)$$
$$+ \frac{(x-0)(x-1)(x-4)(x-5)(x-6)}{(2-0)(2-1)(2-4)(2-5)(2-6)}(15)$$
$$+ \frac{(x-0)(x-1)(x-2)(x-5)(x-6)}{(4-0)(4-1)(4-2)(4-5)(4-6)}(5)$$
$$+ \frac{(x-0)(x-1)(x-2)(x-4)(x-6)}{(5-0)(5-1)(5-2)(5-5)(5-6)}(6)$$
$$+ \frac{(x-0)(x-1)(x-2)(x-4)(x-5)}{(6-0)(6-1)(6-2)(6-4)(6-5)}(19)$$

$$y(x) = \frac{(x-1)(x-2)(x-4)(x-5)(x-6)}{(-1)(-2)(-4)(-5)(-6)}$$
$$+ (14)\frac{(x)(x-2)(x-4)(x-5)(x-6)}{(1)(-1)(-3)(-4)(-5)}$$
$$+ \frac{(x)(x-1)(x-4)(x-5)(x-6)}{(2)(1)(-2)(-3)(-4)}(15)$$
$$+ \frac{(x)(x-1)(x-2)(x-5)(x-6)}{(4)(3)(2)(-1)(-2)}(5)$$
$$+ \frac{(x)(x-1)(x-2)(x-4)(x-6)}{(5)(4)(3)(1)(-1)}(6)$$
$$+ \frac{(x)(x-1)(x-2)(x-4)(x-5)}{6(5)(4)(2)(1)}(19)$$

Substitute $x = 3$ we get

$$y(3) = -\frac{(3-1)(3-2)(3-4)(3-5)(3-6)}{240}$$
$$+ \frac{(3)(3-2)(3-4)(3-5)(3-6)}{60}(14)$$
$$- \frac{(3)(3-1)(3-4)(3-5)(3-6)}{48}(15)$$
$$+ \frac{(3)(3-1)(3-2)(3-5)(3-6)}{48}(5)$$
$$- \frac{(3)(3-1)(3-2)(3-4)(3-6)}{60}(6)$$
$$+ \frac{(3)(3-1)(3-2)(3-4)(3-5)}{240}(19)$$

$$= -\frac{(2)(1)(-1)(-2)(-3)}{240} + \frac{3(1)(-1)(-2)(-3)(14)}{60}$$
$$- \frac{3(2)(-1)(-2)(-3)(15)}{48} + \frac{3(2)(1)(-2)(-3)(5)}{48}$$
$$- \frac{3(2)(1)(-1)(-3)(6)}{60} + \frac{3(2)(1)(-1)(-2)(19)}{240}$$

$$= h\int_0^n \left(y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\right) du$$

[by 1]

$$= h\int_0^n \left(y_0 + u\Delta y_0 + \frac{u^2-u}{2!}\Delta^2 y_0 + \frac{u^3-3u^2+2u}{3!}\Delta^3 y_0 + \ldots\right) du$$

$$= h\left[(uy_0)_0^n + \left(\frac{u^2}{2}\Delta y_0\right)_0^n + \frac{1}{2}\left(\frac{u^3}{3} - \frac{u^2}{2}\right)_0^n \Delta^2 y_0 + \frac{1}{6}\left(\frac{u^4}{4} - u^3 + u^2\right)_0^n \Delta^3 y_0 + \ldots\right]$$

$$= h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \ldots\right]$$

$$\therefore \int_{x_0}^{x_0+nh} y(x)\,dx = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \ldots\right] \quad \ldots (A)$$

This gives the general **Quadrature Formula** for equidistant ordinates and is known as **Newton-Cote's formula**.

TRAPEZOIDAL RULE

Putting $n = 1$ in (A), we get

$$\int_{x_0}^{x_0+h} y(x)\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

(neglecting higher order differences)

$$= \frac{h}{2}[2y_0 + \Delta y_0] = \frac{h}{2}[y_0 + (y_0 + \Delta y_0)] \quad \ldots (1)$$

$$= \frac{h}{2}[y_0 + y_1]$$

In the interval $(x_0 + h, x_0 + 2h)$, we get

$$\int_{x_0+h}^{x_0+2h} y(x)\,dx = h\left[y_1 + \frac{1}{2}\Delta y_1\right]$$

$$= \frac{h}{2}[2y_1 + \Delta y_1] = \frac{h}{2}[y_1 + (y_1 + \Delta y_1)]$$

By **Trapezoidal rule**, we have

$$\int_4^{5.2} \ln x \, dx = \frac{h}{2}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{0.2}{2}[(1.386294 + 1.648658) + 2(1.435084 + 1.481604 + 1.526056 + 1.568616 + 1.609437)]$$

$$= (0.1)[3.034952 + 15.241562]$$

$$\int_4^{5.2} \ln x \, dx = \mathbf{1.8276544}$$

Example 2 : ***Evaluate $\int_0^1 e^{-x^2} dx$ by dividing the range of integration into 4 equal parts using trapezoidal rule.***

Solution : *[Nov. '91, Nov. '9..]*

Here the length of the interval is $h = \frac{1-0}{4} = 0.25$. The values of the function $y = e^{-x^2}$ for each point of subdivision are given below.

x	0	0.25	0.5	0.75	1
e^{-x^2}	1	0.9394	0.7788	0.5698	0.3678
	y_0	y_1	y_2	y_3	y_4

By **Trapezoidal rule** we have

$$\int_0^1 e^{-x^2} dx = \frac{h}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= \frac{0.25}{2}[1.3678 + 2(2.2876)] = (0.125)(5.943)$$

$$\int_0^1 e^{-x^2} dx = \mathbf{0.7428}$$

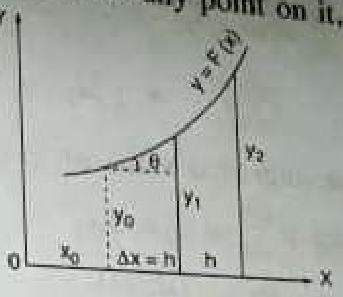
Example 3 : ***Evaluate $\int_0^1 \frac{dx}{1+x^2}$, using Trapezoidal rule with $h = 0.2$. Hence determine the value of π.***

Solution : *[April '92]*

Here $h = 0.2$. The values of the function $y = \frac{1}{1+x^2}$ for each point of subdivision are given below.

x	0	0.2	0.4	0.6	0.8	1
$y = \frac{1}{1+x^2}$	1	0.9615	0.8621	0.7353	0.6098	0.5
	y_0	y_1	y_2	y_3	y_4	y_5

The graph of (2) is a curve in the XY plane; and since a smooth curve ...ctically straight for a short distance from any point on it, we have ... figure



$$\tan\theta \approx \frac{\Delta y}{\Delta x}$$

$$\Delta y \approx \Delta x \tan\theta$$

$$= \Delta x \left(\frac{dy}{dx}\right)_0 \qquad \ldots (3)$$

$$\left[\because \text{Slope at } (x_0, y_0) = \left(\frac{dy}{dx}\right)_0 = \tan\theta\right]$$

$$\therefore y_1 = y_0 + \Delta y$$

$$\therefore y_1 = y_0 + \left(\frac{dy}{dx}\right)_0 \Delta x$$

$$y_1 \approx y_0 + f(x_0, y_0)\, h \qquad [\text{Assuming } \Delta x = h]$$

$$\left[\because \frac{dy}{dx} = f(x, y) \text{ from (1)} \quad \therefore \left(\frac{dy}{dx}\right)_0 = f(x_0, y_0)\right]$$

The next value of y corresponding to $x = x_2 \,(= x_1 + h)$ is

$$y_2 \approx y_1 + \left(\frac{dy}{dx}\right)_1 h$$

$$y_2 \approx y_1 + f(x_1, y_1)\, h \qquad \left[\because \left(\frac{dy}{dx}\right)_1 = f(x_1, y_1)\right]$$

$$y_3 \approx y_2 + f(x_2, y_2)\, h, \text{ etc.}$$

In general $y_{n+1} \approx y_n + f(x_n, y_n)\, h$

By taking h small enough and proceeding in this manner, we could ...culate the expression (2) as a set of corresponding values of x and y. This is **method is given by Euler.** This method is either too slow (in case of h is small) or too inaccurate (in case h is not small) for practical use.

... IMPROVED EULER'S METHOD

Let the given first order differential equation be

$$\frac{dy}{dx} = f(x, y) \qquad \ldots (1)$$

Let us solve this equation under the condition $y(x_0) = y_0$.

Starting with the initial value y_0, an approximate value for y_1 is computed from the relation

$$y_1 \approx y_0 + f(x_0, y_0)\, h \qquad \ldots (2)$$

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In general,

$$y_{n+1} = y_n + h f\left[x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right]$$

(or) $$y(x+h) = y(x) + h f\left[x + \frac{h}{2}, y + \frac{h}{2} f(x, y)\right]$$

This formula is called **modified Euler's formula.**

Note : 1. *The error in the solution can be reduced by decreasing the step size.*

2. *The method will provide error-free solutions if the given function is linear.*

3. *Because in Euler's method we use straight-line segments to approximate the solution, this method is referred to as a first order method.*

4. *The error in one-step of the modified Euler method is $0(h^3)$. This is called the local error. There is an accumulation of error from step to step, so that the error over the whole range is $0(h^2)$ and is called global error.*

Example 1 : *Using Improved Euler's method find y (0.2) and y (0.4) from $y' = x + y, y(0) = 1$ with $h = 0.2$.* *[A.U May 2000]*

Solution :

The Improved Euler's algorithm is

$$y_{m+1} = y_m + \frac{h}{2}\{f(x_m, y_m) + f[x_m + h, y_m + h f(x_m, y_m)]\} \quad ...(1)$$

Putting $m = 0$ in (1) we get

$$y_1 = y_0 + \frac{h}{2}\{f(x_0, y_0) + f[x_0 + h, y_0 + hf(x_0, y_0)]\} \quad ...(2)$$

Here $x_0 = 0,\ y_0 = 1,\ f(x, y) = x + y,\ h = 0.2$

$$\therefore f(x_0, y_0) = x_0 + y_0 = 0 + 1 = 1 \quad ...(3)$$

Substituting (3) in (2) we get

$$\begin{aligned} y_1 &= y_0 + \frac{h}{2}\{1 + f[x_0 + h, y_0 + h \cdot 1]\} \\ &= 1 + 0.1\{1 + f[x_0 + h, y_0 + h]\} \\ &= 1 + 0.1\{1 + f(0 + 0.2, 1 + 0.2)\} \\ &= 1 + 0.1\{1 + f(0.2, 1.2)\} \quad ...(4) \end{aligned}$$

Now $f(0.2, 1.2) = 0.2 + 1.2 = 1.4 \quad ...(5)$

Substituting (5) in (4) we get

$$\begin{aligned} y_1 &= 1 + 0.1\{1 + 1.4\} \\ &= 1 + (0.1)(2.4) = 1 + 0.24 \\ y_1 &= 1.24 \end{aligned}$$

$$\therefore \mathbf{y(0.2) = 1.24}$$

Now $y_2 = y_1 + hy_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \ldots$

$$y_2 = y_1 + hz_1 + \frac{h^2}{2!} z_1' + \frac{h^3}{3!} z_1'' + \ldots \quad \ldots(7)$$

$$z_1' = (x^3 - y^2 z)_1 = x_1^3 - y_1^2 z_1$$
$$= (1.1)^3 - (1.1002)^2 (1.0055)$$
$$= 1.331 - 1.2170 = 0.1140$$
$$z_1'' = (3x^2 - y^2 z' - 2z^2 y)_1$$
$$= 3x_1^2 - y_1^2 z_1' - 2z_1^2 y_1$$
$$= 3(1.1)^2 - (1.1002)^2 (0.1140) - 2(1.0055)^2 (1.1002)$$
$$= 3.63 - 0.1376 - 2.2248$$
$$= 1.2676$$

Substituting these values in (7), we get

$$y_2 = y(1.2) = 1.1002 + (0.1)(1.0055) + \frac{(0.1)^2}{2!}(0.1140) + \frac{(0.1)^3}{3!}(1.2676)$$
$$= 1.1002 + 0.10055 + 0.000570 + 0.000211$$

$y(1.2) =$ **1.2015**

x	1.1	1.2
y	**1.1002**	**1.2015**

❑ EXERCISES ❑

1. Solve $\frac{d^2y}{dx^2} = -xy$ at $x = 0.1, 0.2$ given that $\frac{dy}{dx} = 0.5$, $y = 1$ when $x = 0$.

[Ans. y (0.1) = 1.049, y (0.2) = 1.099]

2. Given $\frac{d^2y}{dx^2} - x\left(\frac{dy}{dx}\right)^2 + y^2 = 0$, with $y(0) = 1$, $y'(0) = 0$, obtain the values of $y(0.1)$ and $y(0.2)$ correct to 3 decimal places using Taylor series method.

[Ans. y (0.1) = 0.995, y (0.2) = 0.981]

3. Solve the equation $\frac{d^2y}{dx^2} + y = 0$ with the conditions $y(0) = 1$ and $y'(0) = 0$. Find $y(0.2)$ and $y(0.4)$.

[Ans. y (0.2) = 1.0204, y (0.4) = 1.0[illegible]]

❑ EULER'S METHOD

Consider the first order differential equation

$$\frac{dy}{dx} = f(x, y) \quad \ldots(1)$$

Let us solve this differential equation under the condition $y(x_0) = y_0$. The solution of (1) gives y as a function x, which may be written symbolically as

$$y = F(x) \quad \ldots(2)$$

$$= 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right] = 2h\left[\frac{6y_0 + 6\Delta y_0 + \Delta^2 y_0}{6}\right]$$

$$= 2h\left[\frac{6y_0 + 6(y_1 - y_0) + y_2 - 2y_1 + y_0}{6}\right]$$

$$= \frac{h}{3}[y_0 + 4y_1 + y_2]$$

$$\therefore \int_{x_0}^{x_0+2h} y(x)\,dx = \frac{h}{3}[y_0 + 4y_1 + y_2] \quad \ldots(1)$$

Similarly for the next two intervals $x_0 + 2h$ to $x_0 + 4h$ we get,

$$\int_{x_0+2h}^{x_0+4h} y(x)\,dx = \frac{h}{3}[y_2 + 4y_3 + y_4] \quad \ldots(2)$$

In general,

$$\int_{x_0+(n-2)h}^{x_0+nh} y(x)\,dx = \frac{h}{3}[y_{n-2} + 4y_{n-1} + y_n] \quad \ldots(3)$$

Adding all the above integrals (1), (2), (3) we get,

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}[y_0 + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 + \ldots) + y_n]$$

$$= \frac{h}{3}[y_0 + y_n + 4\,(\text{sum of odd ordinates}) + 2\,(\text{sum of even ordinates})]$$

This is called Simpson's one third rule or Simpson's $\frac{1}{3}$ rule.

Note 1 : ***When using this formula the student must bear in mind that the interval of integration must be divided into an even number of subintervals of width h.***

Note 2 : ***Simpson's $\frac{1}{3}$ rule is also called a closed formula, since the end points y_0 and y_1 are also included in the formula.***

TRUNCATION ERROR IN SIMPSON'S RULE

The Taylor series expansion of $y = f(x)$ about $x = x_1$ is given by

$$y = y_1 + \frac{(x - x_1)}{1!}y_1' + \frac{(x - x_1)^2}{2!}y_1'' + \ldots \quad \ldots(1)$$

where y_1 is the value of y at $x = x_1$ and $y_1', y_1'', \ldots$ *etc.* are the values of $y', y'', \ldots$ *etc.* at $x = x_1$.

By **Trapezoidal rule** we have,

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\,[(y_0+y_5)+2(y_1+y_2+y_3+y_4)]$$

$$= \frac{0.2}{2}\,[1.5+2(3.1687)] = (0.1)(7.8374)$$

$$\int_0^1 \frac{dx}{1+x^2} = \mathbf{0.78374}$$

We know that

$$\int_0^1 \frac{dx}{1+x^2} = (\tan^{-1} x)_0^1 = \frac{\pi}{4} \qquad \therefore\ \pi = 4\int_0^1 \frac{dx}{1+x^2}$$

$$= 4\,(0.78374) \qquad \text{[From Trapezoidal Rule]}$$

$$\therefore\quad \pi = \mathbf{3.13496}$$

Example 4 : ***Using Trapezoidal rule evaluate*** $\int_{0.6}^{2} y\,dx$ ***from the following table.***

x	**0.6**	**0.8**	**1.0**	**1.2**	**1.4**	**1.6**	**1.8**	**2.0**
y	**1.23**	**1.58**	**2.03**	**4.32**	**6.25**	**8.36**	**10.23**	**12.45**

Solution :

Here $h = 0.2$

x	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
y	1.23	1.58	2.03	4.32	6.25	8.36	10.23	12.45
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7

By Trapezoidal rule, we have

$$\int_{0.6}^{2} y\,dx = \frac{h}{2}\,[(y_0+y_7)+2(y_1+y_2+y_3+y_4+y_5+y_6)]$$

$$= \frac{0.2}{2}\,[13.68+2(1.58+2.03+4.32+6.25+8.36+10.23)]$$

$$= (0.1)\,[79.22]$$

$$\int_{0.6}^{2} y\,dx = \mathbf{7.922}$$

❑ SIMPSON'S $\frac{1}{3}$ RULE

Putting $n = 2$ in the above relation (A) (Refer Pg. No. 4.87) and neglecting all differences above the second we get,

$$\int_{x_0}^{x_0+2h} y(x)\,dx = h\left[2y_0 + \frac{2^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{2^3}{3} - \frac{2^2}{2}\right)\Delta^2 y_0\right]$$

$$= y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \ldots \quad \ldots (4)$$

where $h = x_2 - x_1$

Substituting (4) in (3), we get

$$A_1 = \frac{h}{2}\left[2y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \ldots\right]$$

$$= hy_1 + \frac{h^2}{2!} y_1' + \frac{h^3}{2 \times 2!} y_1'' + \ldots \quad \ldots (5)$$

(2) - (5) $\Rightarrow$

$$\int_{x_1}^{x_2} y\,dx - A_1 = \left(\frac{1}{3!} - \frac{1}{2 \times 2!}\right) h^3 y_1'' + \ldots$$

$$= \frac{-h^3}{12} y_1'' + \ldots$$

Principal part of the error in (x_1, x_2)

$$= \frac{-h^3}{12} y_1''$$

Similarly principal part of the error in the interval (x_2, x_3)

$$= \frac{-h^3}{12} y_2'' \text{ and so on.}$$

Hence the total error $E = \frac{-h^3}{12}\left[y_1'' + y_2'' + \ldots + y_n''\right]$

$$\therefore E < \frac{-nh^3}{12} y''(\xi)$$

Where $y''(\xi)$ is the largest of the n quantities $y_1'', y_2'', \ldots, y_n''$.

$$E < \frac{-nh^3}{12} y''(\xi) = -\frac{(b-a)\,h^2}{12} y''(\xi) \qquad \left[\because n = \frac{b-a}{h}\right]$$

$\therefore$ Error in the trapezoidal rule is of the order h^2.

Example 1 : *Compute the value of the definite integral* $\int_4^{5.2} \log_e x\, dx$ *or* $\int_4^{5.2} \ln x dx$ *using trapezoidal rule.*

Solution :

Divide the interval of integration into six equal parts each of width 0.2 i.e., $h = 0.2$. The values of the function $y = \ln x$ are next calculated for each point of subdivision as given below.

x	4.0	4.2	4.4	4.6	4.8	5.0	5.2
$\ln x$	1.386294	1.435084	1.481604	1.526056	1.568616	1.609437	1.648658
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

9. Given $\frac{dy}{dx} = \frac{x-y}{x+y}$, $y(2) = 1$, Find $y(1)$ by taking $h = -0.2$ using Euler's modified method. *[Ans. $y(1) = 0.73207$]*

10. Given $\frac{dy}{dx} = \frac{y}{x} - \frac{5}{2}x^2 y^3$, $y(1) = \frac{1}{\sqrt{2}}$. Find $y(2)$ with $h = 0.125$ using Euler modified method. *[Ans. $y(2) = 0.34939$]*

11. Solve $\frac{dy}{dx} = x\sqrt{1+y^2}$, $y(1) = 0$ at $x = 3$ taking $h = 0.4$ using Euler's modified method. *[Ans. $y(3) = 21.671$]*

❑ THE RUNGE - KUTTA METHODS

This method was devised by Runge, about the year 1934 and extended by Kutta a few years later. Therefore we call this method as **Runge-Kutta method**. Unlike any of the methods, explained in the preceding two sections the increments of the function are calculated once for all by means of a definite set of formulae.

Here a set of formulae are given without proof for solving a differential equation of the form $\frac{dy}{dx} = f(x, y)$ under the initial condition $y(x_0) = y_0$. Let h denote the length of the interval between equidistant values of x. The various types of formulae according to their order are given below.

❑ SECOND ORDER RUNGE - KUTTA METHOD

If the initial values are x_0, y_0 for the differential equation $\frac{dy}{dx} = f(x, y)$ then the first increment in y *viz* Δy is computed from the formulae

$$k_1 = h f(x_0, y_0)$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$\Delta y = k_2$$

$$\text{Now } x_1 = x_0 + h, \; y_1 = y_0 + \Delta y$$

The increment in y for the second interval is computed in a similar manner by means of the formulae.

$$k_1 = h f(x_1, y_1),$$

$$k_2 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right),$$

$$\Delta y = k_2.$$

and so on for the succeeding intervals.

$$y_1 = y_0 + \Delta y$$
$$= 1 + 0.24135$$
$$y(1.1) = \mathbf{1.24135}$$

To find $y(1.1)$ using Runge-Kutta Method of Fourth order :

Here $x_0 = 1$, $y_0 = 1$ and $h = 0.1$

Now $k_1 = hf(x_0, y_0)$

$$= h\,(y_0^2 + x_0 y_0)$$
$$= (0.1)\,[1^2 + (1)(1)] = (0.1)(2)$$
$$k_1 = 0.2$$
$$k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right]$$
$$= h\left[\left(y_0 + \frac{k_1}{2}\right)^2 + \left(x_0 + \frac{h}{2}\right)\left(y_0 + \frac{k_1}{2}\right)\right]$$
$$= (0.1)\left[\left(1 + \frac{0.2}{2}\right)^2 + \left(1 + \frac{0.1}{2}\right)\left(1 + \frac{(0.2)}{2}\right)\right]$$
$$= (0.1)\,[(1.1)^2 + (1.05)(1.1)]$$
$$k_2 = 0.2365$$
$$k_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right]$$
$$= h\left[\left(y_0 + \frac{k_2}{2}\right)^2 + \left(x_0 + \frac{h}{2}\right)\left(y_0 + \frac{k_2}{2}\right)\right]$$
$$= (0.1)\left[\left(1 + \frac{0.2365}{2}\right)^2 + \left(1 + \frac{0.1}{2}\right)\left(1 + \frac{0.2365}{2}\right)\right]$$
$$= (0.1)\,[(1.11825)^2 + (1 + 0.05)(1.11825)]$$
$$= (0.1)\,[2.4246]$$
$$k_3 = 0.24246$$
$$k_4 = hf(x_0 + h, y_0 + k_3)$$
$$= (0.1)\,[(y_0 + k_3)^2 + (x_0 + h)(y_0 + k_3)]$$
$$= (0.1)\,[(1.24246)^2 + (1.1)(1.24246)]$$
$$= (0.1)\,[1.5437 + 1.366706] = (0.1)\,[2.91041]$$
$$k_4 = 0.29104$$
$$\therefore \Delta y = \frac{1}{6}\,[k_1 + 2k_2 + 2k_3 + k_4]$$
$$= \frac{1}{6}\,[0.2 + 2(0.2365) + 2(0.24246) + 0.29104]$$
$$= \frac{1}{6}\,[1.44896] = 0.24149$$
$$\therefore y_1 = y_0 + \Delta y = 1 + 0.24149 = 1.24149$$
$$y(0.1) = \mathbf{1.24149}$$

$$y_3 = y_2 + \Delta y$$
$$y(0.3) = y(0.2) + \Delta y$$
$$= 1.2427 + 0.1569$$
$$\therefore y(0.3) = 1.3996$$

x	0	0.1	0.2	0.3
y	1	1.1103	1.2427	1.3996

Example 2 : ***Find the values of y(1.1) using Runge-Kutta method of the third order and (ii) Runge-Kutta method of the fourth order given that*** $\frac{dy}{dx} = y^2 + xy \,;\, y(1) = 1$.

Solution :

Given
$$y' = y^2 + xy$$
$$f(x, y) = y^2 + xy$$

And also given that $x_0 = 1$, $y_0 = 1$ and $h = 0.1$

To find $y(1.1)$ using third order Runge-Kutta Method :

Now
$$k_1 = hf(x_0, y_0)$$
$$= h[y_0^2 + x_0 y_0]$$
$$= (0.1)[(1)^2 + (1)(1)] = (0.1)(2)$$
$$k_1 = 0.2$$
$$k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right]$$
$$= h\left[\left(y_0 + \frac{k_1}{2}\right)^2 + \left(x_0 + \frac{h}{2}\right)\left(y_0 + \frac{k_1}{2}\right)\right]$$
$$= (0.1)\left[\left(1 + \frac{0.2}{2}\right)^2 + \left(1 + \frac{0.1}{2}\right)\left(1 + \frac{0.2}{2}\right)\right]$$
$$= (0.1)[(1 + 0.1)^2 + (1 + 0.05)(1 + 0.1)]$$
$$k_2 = 0.2365$$
$$k_3 = hf(x_0 + h, y_0 + 2k_2 - k_1)$$
$$= h[(y_0 + 2k_2 - k_1)^2 + (x_0 + h)(y_0 + 2k_2 - k_1)]$$
$$= (0.1)\{[(1 + 2(0.2365) - 0.2]^2 + (1 + 0.1)[1 + 2(0.2365) - 0.2]\}$$
$$= (0.1)[(1.273)^2 + (1.1)(1.273)]$$
$$= (0.1)[3.0208]$$
$$k_3 = 0.30208$$
$$\therefore \quad \Delta y = \frac{1}{6}[k_1 + 4k_2 + k_3]$$
$$= \frac{1}{6}[0.2 + 4(0.2365) + 0.30208] = \frac{1}{6}[1.44808]$$
$$\therefore \Delta y = 0.24135$$

$$k_3 = hf(x_0 + h, y_0 + 2k_2 - k_1)$$
$$= h(x_0 + h + y_0 + 2k_2 - k_1)$$
$$= (0.1)[0.1 + 0.1 + 1.1103 + 2(0.13208) - 0.12103]$$
$$= (0.1)(1.4534)$$
$$k_3 = 0.14534$$
$$\therefore \Delta y = \frac{1}{6}[k_1 + 4k_2 + k_3]$$
$$= \frac{1}{6}[0.12103 + 4(0.13208) + 0.14534]$$
$$= \frac{1}{6}(0.7947)$$
$$\therefore \Delta y = 0.1324$$
$$y_2 = y_1 + \Delta y$$
$$y(0.2) = y(0.1) + \Delta y$$
$$= 1.1103 + 0.1324$$
$$\therefore \mathbf{y(0.2) = 1.2427}$$

To find y (0.3) using third order Runge-Kutta Method :

Here $x_0 = 0.2,\ y_0 = 1.2427,\ h = 0.1$

Now $k_1 = hf(x_0, y_0) = h(x_0 + y_0)$
$$= (0.1)[0.2 + 1.2427] = (0.1)(1.4427)$$
$$k_1 = 0.14427$$
$$k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right]$$
$$= h\left[x_0 + \frac{h}{2} + y_0 + \frac{k_1}{2}\right]$$
$$= (0.1)\left[0.2 + \frac{0.1}{2} + 1.2427 + \frac{0.14427}{2}\right]$$
$$= (0.1)(1.5648)$$
$$k_2 = 0.15648$$
$$k_3 = hf(x_0 + h, y_0 + 2k_2 - k_1)$$
$$= h(x_0 + h + y_0 + 2k_2 - k_1)$$
$$= (0.1)[0.2 + 0.1 + 1.2427 + 2(0.15648) - 0.14427]$$
$$= (0.1)(1.7114)$$
$$k_3 = 0.17114$$
$$\therefore \Delta y = \frac{1}{6}[k_1 + 4k_2 + k_3]$$
$$= \frac{1}{6}[0.14427 + 4(0.15648) + 0.17114]$$
$$= \frac{1}{6}(0.9413)$$
$$\therefore \Delta y = 0.1569$$

The increment in y for the second interval is computed in a similar manner by means of the formulae.

$$k_1 = h f(x_1, y_1),$$

$$k_2 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right),$$

$$k_3 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right),$$

$$k_4 = h f(x_1 + h, y_1 + k_3),$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

and so on for the succeeding intervals.

Note : **If $\frac{dy}{dx}$ is a function of x alone then the Fourth Order Runge-Kutta method reduces to Simpson's Rule.**

For, $\quad k_1 = hf(x_0)$

$$k_2 = hf\left(x_0 + \frac{h}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}\right)$$

$$k_4 = hf(x_0 + h)$$

$$\therefore \quad \Delta y = \frac{h}{6}\left[f(x_0) + 2f\left(x_0 + \frac{h}{2}\right) + 2f\left(x_0 + \frac{h}{2}\right) + f(x_0 + h)\right]$$

$$= \frac{\left(\frac{h}{2}\right)}{3}\left[f(x_0) + 4f\left(x_0 + \frac{h}{2}\right) + f(x_0 + h)\right]$$

which is the same result as would be obtained by applying Simpson's Rule in the interval x_0 to $x_0 + h$ if we take two equal sub-intervals of width $\frac{h}{2}$.

Now we shall illustrate this method in the following examples.

Example 1 : ***Use Runge – Kutta method to approximate y, when $x = 0.1, 0.2, 0.3$, $h = 0.1$ given $x = 0$ when $y = 1$ and $\frac{dy}{dx} = x + y$***

Solution : *[A.U. Nov. '91]*

Given

$$y' = x + y$$

$$\text{i.e., } f(x, y) = x + y$$

And also given that $x_0 = 0$, $y_0 = 1$ and $h = 0.1$

To find $y(0.1)$ using third order Runge-Kutta Method :

Now $k_1 = hf(x_0, y_0)$

$= h[x_0 + y_0]$

$k_1 = (0.1)[0+1] = 0.1$

$$k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right]$$

$$= h\left[x_0 + \frac{h}{2} + y_0 + \frac{k_1}{2}\right]$$

$$= (0.1)\left[0 + \frac{0.1}{2} + 1 + \frac{0.1}{2}\right]$$

$= (0.1)[0.05 + 1.05]$

$k_2 = 0.11$

$k_3 = hf(x_0 + h, y_0 + 2k_2 - k_1)$

$= h(x_0 + h + y_0 + 2k_2 - k_1)$

$= (0.1)\ [0 + 0.1 + 1 + 2(0.11) - 0.1]$

$= (0.1)(1.22)$

$k_3 = 0.122$

$$\therefore \Delta y = \frac{1}{6}[k_1 + 4k_2 + k_3]$$

$$= \frac{1}{6}[0.1 + 4(0.11) + 0.122] = \frac{1}{6}(0.662)$$

$\therefore \Delta y = 0.1103$

$y_1 = y_0 + \Delta y$

$= 1 + 0.1103$

$\therefore$ **$y(0.1) = 1.1103$**

To find $y(0.2)$ using third order Runge-Kutta Method :

Here $x_0 = 0.1,\ y_0 = 1.1103,\ h = 0.1$

Now $k_1 = hf(x_0, y_0) = h(x_0 + y_0)$

$= (0.1)[0.1 + 1.1103]$

$k_1 = 0.12103$

$$k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right]$$

$$= h\left[x_0 + \frac{h}{2} + y_0 + \frac{k_1}{2}\right]$$

$$= (0.1)\left[0.1 + \frac{0.1}{2} + 1.1103 + \frac{0.12103}{2}\right]$$

$= (0.1)(1.3208)$

$k_2 = 0.13208$

...ORDER RUNGE - KUTTA METHOD

...d order Runge-Kutta method is designed by the following

$$k_1 = h f(x_0, y_0),$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right),$$

$$k_3 = h f(x_0 + h, y_0 + 2k_2 - k_1),$$

...e first increment in y viz Δy is computed from

$$\Delta y = \frac{1}{6}[k_1 + 4k_2 + k_3]$$

$$\text{Now } x_1 = x_0 + h, \; y_1 = y_0 + \Delta y$$

...crement in y for the second interval is computed in a similar ...y means of the formulae.

$$k_1 = h f(x_1, y_1),$$

$$k_2 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right),$$

$$k_3 = h f(x_1 + h, y_1 + 2k_2 - k_1),$$

$$\Delta y = \frac{1}{6}[k_1 + 4k_2 + k_3].$$

...on for the succeeding intervals.

...RTH ORDER RUNGE-KUTTA METHOD

...ethod is most commonly used in practice. Unless and otherwise ...unge-Kutta method means only Fourth Order Runge-Kutta ...

Let $\frac{dy}{dx} = f(x, y)$ be a given differential equation to be solved ...e condition $y(x_0) = y_0$. If h be the length of the interval between ...nt values, then the first increment in y is computed from the ...

$$k_1 = h f(x_0, y_0),$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right),$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = h f(x_0 + h, y_0 + k_3)$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$\text{Now } x_1 = x_0 + h, \; y_1 = y_0 + \Delta y$$

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$$f[x_1 + h, y_1 + h f(x_1, y_1)] = f(0.4, 0.53485)$$
$$= 0.53485 + e^{0.4}$$
$$= 2.02667 \quad \ldots (9)$$

Substituting (7) and (9) in (6), we get

$$y_2 = y(0.4) = 0.24214 + \frac{0.2}{2}[1.46354 + 2.02667]$$
$$= 0.59116$$
$$\therefore \mathbf{y(0.4) = 0.59116}$$

x	0	0.2	0.4
y	0	0.24214	0.59116

Example 5 : *Given $y' = x^2 - y$, $y(0) = 1$, find correct to four decimal places the value of $y(0.1)$, by using Improved Euler's Method.*

Solution :

The Improved Euler's algorithm is

$$y_{m+1} = y_m + \frac{h}{2}\{f(x_m, y_m) + f[x_m + h, y_m + h f(x_m, y_m)]\} \quad \ldots (1)$$

Putting $m = 0$ in (1) we get

$$y_1 = y_0 + \frac{h}{2}\{f(x_0, y_0) + f[x_0 + h, y_0 + hf(x_0, y_0)]\} \quad \ldots (2)$$

Here $x_0 = 0$, $y_0 = 1$, and $h = 0.1$; $f(x, y) = x^2 - y$

$$\therefore f(x_0, y_0) = x_0^2 - y_0 = (0)^2 - 1 = -1 \quad \ldots (3)$$

Substituting (3) in (2), we get

$$y_1 = 1 + \frac{0.1}{2}\{-1 + f[0 + 0.1, 1 + 0.1(-1)]\}$$
$$= 1 + 0.05\{(-1) + f(0.1, 0.9)\} \quad \ldots (4)$$

Now $f(0.1, 0.9) = (0.1)^2 - 0.9 = 0.01 - 0.9 = -0.89 \quad \ldots (5)$

Substituting (5) in (4), we get

$$y_1 = 1 + 0.05\{(-1) - 0.89\}$$
$$= 0.9055$$
$$\therefore \mathbf{y(0.1) = 0.9055}$$

Example 6 : *Find the values of $y(1.2)$ and $y(1.4)$, using Improved Euler's method with $h = 0.2$, given that $\frac{dy}{dx} = \frac{2y}{x} + x^3$; $y(1) = 0.5$.*

Solution :

The Improved Euler's algorithm is

$$y_{m+1} = y_m + \frac{h}{2}\{f(x_m, y_m) + f[x_m + h, y_m + h f(x_m, y_m)]\} \quad \ldots (1)$$

Putting $m = 0$ in (1), we get

$$y_1 = y_0 + \frac{h}{2}\{f(x_0, y_0) + f[x_0 + h, y_0 + hf(x_0, y_0)]\} \quad \ldots (2)$$

$\therefore f(0.2, 1.1872) = 1.1872 - \frac{2(0.2)}{1.1872}$

$= 0.8503$ ... (9)

Substituting (9) in (8) we get

$y_2 = 1.0959 + 0.05\{0.9135 + 0.8503\}$

$y(0.2) = \mathbf{1.1841}$

x	0	0.1	0.2
y	1	1.0959	1.1841

Example 4 : ***Solve $\frac{dy}{dx} = y + e^x$, $y(0) = 0$ for $x = 0.2, 0.4$ by using improved Euler's method.***

Solution:

Given $\frac{dy}{dx} = y + e^x$; $x_0 = 0$, $y_0 = 0$ and $h = 0.2$

The improved Euler's algorithm is

$$y_{m+1} = y_m + \frac{h}{2}\{f(x_m, y_m) + f[x_m + h, y_m + hf(x_m, y_m)]\} \quad \text{... (1)}$$

Putting $m = 0$ in (1), we get

$$y_1 = y_0 + \frac{h}{2}\{f(x_0, y_0) + f[x_0 + h, y_0 + hf(x_0, y_0)]\} \quad \text{... (2)}$$

Here $x_0 = 0, y_0 = 0$

$f(x, y) = y + e^x$

$f(x_0, y_0) = 0 + e^0 = 1$ ... (3)

Substituting (3) in (2), we get

$$y_1 = y_0 + \frac{h}{2}\{1 + f(x_0 + h, y_0 + h)\}$$

$$= 0 + \frac{0.2}{2}\{1 + f(0 + 0.2, 0 + 0.2)\} \quad \text{... (4)}$$

Now $f(0.2, 0.2) = 0.2 + e^{0.2}$ ... (5)

Substituting (5) in (4), we get

$y_1 = y(0.2) = 0.1[1 + 0.2 + e^{0.2}]$

$= 0.1[1.2 + 1.2214]$

$y_1 = \mathbf{0.24214}$

Putting $m = 1$ in (1), we get

$$y_2 = y_1 + \frac{h}{2}\{f(x_1, y_1) + f[x_1 + h, y_1 + hf(x_1, y_1)]\} \quad \text{... (6)}$$

Here $x_1 = 0.2$, $y_1 = 0.24214$ and $h = 0.2$

Now $f(x_1, y_1) = y_1 + e^{x_1}$

$f(0.2, 0.24214) = 0.24214 + e^{0.2}$ ... (7)

$= 1.46354$

$y_1 + hf(x_1, y_1) = 0.24214 + (0.2)(1.46354)$ ... (8)

$= 0.53485$

Example 3 : ***Using Improved Euler's method find y at x = 0.1 and x = 0.2 given*** $\frac{dy}{dx} = y - \frac{2x}{y}$, $y(0) = 1$. [Apr. 91, Nov. 96]

Solution :

The Improved Euler's algorithm is

$$y_{m+1} = y_m + \frac{h}{2} \{f(x_m, y_m) + f[x_m + h, y_m + h\, f(x_m, y_m)]\} \quad \ldots(1)$$

Putting $m = 0$ in (1) we get

$$y_1 = y_0 + \frac{h}{2} \{f(x_0, y_0) + f[x_0 + h, y_0 + hf(x_0, y_0)]\} \quad \ldots(2)$$

Here $x_0 = 0$, $y_0 = 1$, $f(x, y) = y - \frac{2x}{y}$

$$\therefore f(x_0, y_0) = y_0 - \frac{2x_0}{y_0} = 1 - 0 = 1 \quad \ldots(3)$$

Substituting (3) in (2) we get

$$y_1 = y_0 + \frac{h}{2} \{1 + f[x_0 + h, y_0 + h \cdot 1]\}$$

$$= 1 + \frac{0.1}{2} \{1 + f(0 + 0.1, 1 + 0.1)\}$$

$$= 1 + \frac{0.1}{2} \{1 + f(0.1, 1.1)\} \quad \ldots(4)$$

Now $f(0.1, 1.1) = 1.1 - \frac{2(0.1)}{1.1} = 0.9182 \quad \ldots(5)$

Substituting (5) in (4) we get

$$y_1 = 1 + \frac{0.1}{2}(1 + 0.9182)$$

$$\therefore \mathbf{y(0.1) = 1.0959}$$

Putting $m = 1$ in (1) we get

$$y_2 = y_1 + \frac{h}{2} \{f(x_1, y_1) + f[x_1 + h, y_1 + h\, f(x_1, y_1)]\} \quad \ldots(6)$$

Here $x_1 = 0.1$, $y_1 = 1.0959$

Now $f(x_1, y_1) = f(0.1, 1.0959)$

$$= 1.0959 - \frac{2(0.1)}{1.0959}$$

$$\mathbf{f(x_1, y_1) = 0.9135} \quad \ldots(7)$$

Substituting (7) in (6) we get

$$y_2 = y_1 + \frac{h}{2} \{0.9135 + f[0.2, y_1 + h(0.9135)]\}$$

$$= 1.0959 + \frac{0.1}{2} \{0.9135 + f(0.2, 1.0959 + 0.1(0.9135))\}$$

$$y_2 = 1.0959 + 0.05 \{0.9135 + f(0.2, 1.1872)\} \quad \ldots(8)$$

Putting $m = 1$ in (1) we get

$$y_2 = y_1 + \frac{h}{2}\{f(x_1, y_1) + f[x_1 + h, y_1 + h f(x_1, y_1)]\} \quad \ldots (6)$$

Here $x_1 = 0.2, y_1 = 1.24, h = 0.2$

$$f(x_1, y_1) = f(0.2, 1.24) = 0.2 + 1.24$$

Now $f(x_1, y_1) = 1.44$ ... (7)

Substituting (7) in (6) we get

$$y_2 = 1.24 + \frac{0.2}{2}\{1.44 + f[0.2 + 0.2\ , 1.24 + 0.2(1.44)]\}$$

$$y_2 = 1.24 + 0.1\{1.44 + f(0.4, 1.528)\} \quad \ldots (8)$$

Now $f(0.4, 1.528) = 0.4 + 1.528$

$f(0.4, 1.528) = 1.928$ ... (9)

Substituting (9) in (8) we get

$$y_2 = 1.24 + 0.1(1.44 + 1.928)$$

$$= 1.24 + 0.1(3.368)$$

$$y_2 = 1.24 + 0.3368 = 1.5768$$

$$\therefore \mathbf{y(0.4) = 1.5768}$$

Example 2: ***Using Improved Euler's method solve $y' = x + y + xy$, $y(0) = 1$ compute y at $x = 0.1$, by taking $h = 0.1$.*** *[A.U May '99]*

Solution :

Given $y' = x + y + xy$, $x_0 = 0$, $y_0 = 1$ and $h = 0.1$

The Improved Euler's algorithm is

$$y_{m+1} = y_m + \frac{h}{2}\{f(x_m, y_m) + f[x_m + h, y_m + h f(x_m, y_m)]\} \quad \ldots (1)$$

Putting $m = 0$ in (1) we get

$$y_1 = y_0 + \frac{h}{2}\{f(x_0, y_0) + f[x_0 + h, y_0 + hf(x_0, y_0)]\} \quad \ldots (2)$$

Here $x_0 = 0$, $y_0 = 1$, $f(x, y) = x + y + xy$

$$\therefore f(x_0, y_0) = x_0 + y_0 + x_0 y_0 = 0 + 1 + 0(1) = 1 \quad \ldots (3)$$

Substituting (3) in (2) we get

$$y_1 = 1 + \frac{0.1}{2}\{1 + f[0 + 0.1, 1 + 0.1(1)]\} \quad \ldots (4)$$

$$y_1 = 1 + 0.05\{1 + f(0.1, 1.1)\}$$

Now $f(0.1, 1.1) = 0.1 + 1.1 + (0.1)(1.1) = 1.31$ ... (5)

Substituting (5) in (4) we get

$$y_1 = 1 + 0.05(1 + 1.31)$$

$$y(0.1) = 1 + (0.05)(2.31) = 1 + 0.1155$$

$$= 1.1155$$

$$\therefore \mathbf{y(0.1) = 1.1155}$$

$$= \frac{h}{2} [y_1 + y_2]$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \text{ etc}$$

$$\int_{x_0+(n-1)h}^{x_0+nh} y(x)\,dx = \frac{h}{2} [y_{n-1} + y_n] \quad \ldots(3)$$

Adding (1), (2) and (3), we get

$$\int_{x_0}^{x_0+nh} y(x)\,dx = \frac{h}{2} [(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})] \quad \ldots(I)$$

This is called the **Trapezoidal Rule**.

Note : *The trapezoidal rule is the simplest of the formulae for numerical integration, but it is also the least accurate. The accuracy of the result can be improved by decreasing the interval h.*

❑ TRUNCATION ERROR IN THE TRAPEZOIDAL RULE

The Taylor series expansion of $y = f(x)$ about $x = x_1$ is given by

$$y = y_1 + \frac{(x - x_1)}{1!} y_1' + \frac{(x - x_1)^2}{2!} y_1'' + \ldots \quad \ldots(1)$$

where y_1 is the value of y at $x = x_1$ and $y_1', y_1'' \ldots$ etc are the values of y', y'', etc at $x = x_1$.

$$\therefore \int_{x_1}^{x_2} y\,dx = \int_{x_1}^{x_2} \left[y_1 + \frac{(x - x_1)}{1!} y_1' + \frac{(x - x_1)^2}{2!} y_1'' + \ldots \right] dx$$

$$= \left[y_1 x + \frac{(x - x_1)^2}{2!} y_1' + \frac{(x - x_1)^3}{3!} y_1'' + \ldots \right]_{x_1}^{x_2}$$

$$= y_1 (x_2 - x_1) + \frac{(x_2 - x_1)^2}{2!} y_1' + \frac{(x_2 - x_1)^3}{3} y_1'' + \ldots$$

$$= h y_1 + \frac{h^2}{2!} y_1' + \frac{h^3}{3!} y_1'' + \quad \ldots(2)$$

where $h = x_2 - x_1$

Now, A_1 = area of the trapezium in the interval (x_1, x_2)

$$= \frac{1}{2} h (y_1 + y_2) \quad \ldots(3)$$

Putting $x = x_2$ and $y = y_2$ in (1), we get

$$y_2 = y_1 + \frac{(x_2 - x_1)}{1!} y_1' + \frac{(x_2 - x_1)^2}{2!} y_1'' + \ldots$$

But in engineering problems we frequently come across the integrals whose integrand is an empirical function given by a table. In these cases we may use a numerical method for approximate integration. When we apply numerical integration to a function of a single variable, the process is sometimes called **mechanical quadrature;** when we apply numerical integration to the computation of a double integral involving a function of two independent variables it is called **mechanical cubature.**

The problem of numerical integration, like that of numerical differentiation, is solved by representing the integrand by an interpolation formula and then integrating this formula between the given limits. Thus to find the value of the definite integral $\int_a^b f(x)\,dx$ (or) $\int_a^b y\,dx$ we replace the function $f(x)$ (or y) by an interpolation formula, usually one involving differences, and then integrate this formula between the limits a and b. In this way we can derive quadrature formulae for the approximate integration of any function for which numerical values are known.

Of the many possible quadrature formulae, here we shall derive some of the simplest and most useful one.

❑ Quadrature Formula for Equidistant Ordinates

Consider the Newton's forward difference formula

$$y(x) = y(x_0 + nh) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \ldots$$

This formula can also be written by replacing n by u as

$$y(x) = y(x_0 + uh) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots \quad \ldots (1)$$

Let $y = y(x) \ldots (2)$ be the given function.

Let us now integrate (2) over n equidistant intervals of width $h (= \Delta x)$.

$$\int_{x_0}^{x_0+nh} y(x)\,dx = ?$$

$$\text{Let } x = x_0 + uh$$

$$\therefore dx = h\,du$$

When

$$x = x_0, \quad u = 0$$

$$x = x_0 + nh, \quad u = n$$

$$\therefore \int_{x_0}^{x_0+nh} y(x)\,dx = h\int_0^n y(x_0 + uh)\,du$$

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11. Estimate the annual rate of cloth sales of 1935 from the following data.

Year	Sale of cloth of lakhs of metres
1920	250
1925	285
1930	328
1940	444

[Ans. 11.55]

12. Find the numerical value of the first derivative at $x = 0.4$ of the function defined as under.

x	0.1	0.2	0.3	0.4
$f(x)$	1.10517	1.22140	1.34986	1.49182

[Ans. $f'(0.4) = 4.49134$]

13. Given the following tabulated values of function, prepare a difference table and evaluate y' at $x = 1.2$ *[A.M.I.E. Sep. 88]*

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.025 0

[Ans. 3.3205]

14. From the following data find $\dfrac{dy}{dx}$ at $x = 0.09$.

x	0	0.05	0.10	0.15	0.20	2.0
y	0.0000	0.1001	0.2013	0.3045	0.4107	0.5211

[Ans. 1.99999]

15. Find the first and second derivatives of the function $y = f(x)$ tabulated below at the point $x = 1.1$

x	1.0	1.2	1.4	1.6	1.8	2.0
$f(x)$	0.00	0.1280	0.5440	1.2960	2.4320	4.00

[Ans. 0.630; 6.6]

NUMERICAL INTEGRATION

The term Numerical integration is the numerical evaluation of a definite integral

$$A = \int_a^b f(x)\,dx$$

where 'a' and 'b' are given constants and $f(x)$ is a function given analytically by a formula or empirically by a table of values. Geometrically, A is the area under the curve of $f(x)$ between the ordinates $x = a$ and $x = b$.

We know that Lagrange's formula is

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$y(x) = \frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)}(-12) + \frac{(x-0)(x-3)(x-4)}{(1-0)(1-3)(1-4)}(0) + \frac{(x-0)(x-1)(x-4)}{(3-0)(3-1)(3-4)}(6) + \frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)}(12)$$

$$y(x) = \frac{(x-1)(x-3)(x-4)(-12)}{(-1)(-3)(-4)} + \frac{x(x-1)(x-4)}{(3)(2)(-1)}(6) + \frac{(x)(x-1)(x-3)}{(4)(3)(1)}(12)$$

$$= (x-1)(x-3)(x-4) - x(x-1)(x-4) + x(x-1)(x-3)$$

$$= (x-1)[x^2 - 3x - 4x + 12 - x^2 + 4x + x^2 - 3x]$$

$$= (x-1)(x^2 - 6x + 12)$$

$$= x^3 - 6x^2 + 12x - x^2 + 6x - 12 \quad \ldots (1)$$

$$y(x) = x^3 - 7x^2 + 18x - 12$$

Substituting $x = 2$ in (1), we get

$$y(2) = 2^3 - 7(2)^2 + 18(2) - 12$$

$$= 8 - 28 + 36 - 12 = 44 - 40$$

$$y(2) = 4$$

Example 4 : ***Using Lagrange's interpolation find the polynomial through (0, 0), (1, 1) and (2, 2)*** ***[A.U. May/June 2007]***

Solution :

The given points can be arranged in the form of table as given below.

x	0	1	2
y	0	1	2

Here $x_0 = 0,\quad x_1 = 1,\quad x_2 = 2$

$y_0 = 0,\quad y_1 = 1,\quad y_2 = 2$

$$= \frac{12}{240} - \frac{252}{60} + \frac{540}{48} + \frac{180}{48} - \frac{108}{60} + \frac{228}{240}$$

$$= \frac{12 - 1008 + 2700 + 900 - 432 + 228}{240} = \frac{2400}{240}$$

$\therefore y(3) = 10$

Example 2 : ***Using Lagrange's interpolation formula, find the corresponding to $x = 10$ from the following table :***

x	5	6	9	11
y	12	13	14	16

Solution :

The Lagrange's interpolation formula is

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

Here $x = 10$, $x_0 = 5$, $x_1 = 6$, $x_2 = 9$, $x_3 = 11$

$y_0 = 12$, $y_1 = 13$, $y_2 = 14$ $y_3 = 16$

$$f(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}(12) + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}(13) + \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}(14) + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}(16)$$

$$= \frac{4 \cdot 1 \cdot 1}{1 \cdot 4 \cdot 6}(12) - \frac{5 \cdot 1 \cdot 1}{1 \cdot 3 \cdot 5}(13) + \frac{5 \cdot 4 \cdot 1}{4 \cdot 3 \cdot 2}(14) + \frac{5 \cdot 4 \cdot 1}{6 \cdot 5 \cdot 2}(16)$$

$f(10) = 14.63$

Example 3 : ***Using Lagrange's formula, fit a polynomial to the data.***

x	0	1	3	4
y	−12	0	6	12

also find y at $x = 2$ *[A.U. Nov., Dec. 200*

Solution :

Here $x_0 = 0$, $x_1 = 1$, $x_2 = 3$, $x_3 = 4$

$y_0 = -12$, $y_1 = 0$, $y_2 = 6$, $y_3 = 12$

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Substituting (2), (3), (4), (5) in (1), we get

$$f(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} f(x_0) + \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)} f(x_1) + \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})} f(x_n)$$

If we denote $f(x_0), f(x_1), \dots f(x_n)$ by $y_0, y_1, \dots, y_n$ we get,

$$f(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)} y_1 + \dots + \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})} y_n$$

which is **Lagrange's interpolation** formula.

***Note 1 :* When the arguments $x_0, x_1, \dots, x_n$ are not equally spaced then we can use this formula to find y for any x.**

Example 1 : ***Using Lagrange's formula to calculate f (3) from the following table.*** [A.U. Nov., Dec. 20..]

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

Solution :

Here $x_0 = 0, \quad x_1 = 1, \quad x_2 = 2, \quad x_3 = 4, \quad x_4 = 5, x_5 = 6$

$y_0 = 1, \quad y_1 = 14, y_2 = 15, \quad y_3 = 5, \quad y_4 = 6, y_5 = 19$

We know that Lagrange's formula is

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)} y_3 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_5)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)(x_4-x_5)} y_4 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)} y_5$$

CHAPTER 4

Interpolation, Numerical Differentiation And Numerical Integration

❑ INTERPOLATION WITH UNEQUAL INTERVALS ❑

❑ Lagrange's Interpolation Formula for unequal intervals

Let $f(x_0), f(x_1), \ldots, f(x_n)$ be the values of the function $y = f(x)$ corresponding to the arguments $x_0, x_1, \ldots, x_n$, not necessarily equally spaced.

Let $f(x)$ be a polynomial in x of degree n. Then we can represent $f(x)$ as

$$\begin{aligned} f(x) &= a_0 (x - x_1)(x - x_2) \ldots (x - x_n) \\ &\quad + a_1 (x - x_0)(x - x_2) \ldots (x - x_n) + \ldots \\ &\quad + a_n (x - x_0)(x - x_1) \ldots (x - x_{n-1}) \end{aligned} \qquad \ldots (1)$$

where $a_0, a_1, \ldots, a_n$ are constants.

Now we have to determine the $(n + 1)$ constants $a_0, a_1, \ldots a_n$.

Putting $x = x_0$ in (1), we get

$$f(x_0) = a_0 (x_0 - x_1)(x_0 - x_2) \ldots (x_0 - x_n)$$

$$a_0 = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2) \ldots (x_0 - x_n)} \qquad \ldots (2)$$

Putting $x = x_1$ in (1), we get

$$f(x_1) = a_1 (x_1 - x_0)(x_1 - x_2) \ldots (x_1 - x_n)$$

$$a_1 = \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2) \ldots (x_1 - x_n)} \qquad \ldots (3)$$

Similarly

$$a_2 = \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1) \ldots (x_2 - x_n)} \qquad \ldots (4)$$

....

$$a_n = \frac{f(x_n)}{(x_n - x_0)(x_n - x_1) \ldots (x_n - x_{n-1})} \qquad \ldots (5)$$

Solution :

The difference table is as given below.

x	$y = f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
1	1				
		7			
2	8		12		
		19		6	
3	27		18		
		37		6	0
4	64		24		
		61		6	0
5	125		30		
		91		6	0
6	216		36		
		127		6 ($\nabla^3 x_0$)	0 ($\nabla^4 x_0$)
7	343		42 ($\nabla^2 x_0$)		
		167 (∇x_0)			
8 (x_0)	512 (y_0)				

Here $x_0 = 8$, $y_0 = 512$, $h = 1$ and $x = 7.5$.

Newton's Backward interpolation formula is

$$y(x_0 + nh) = y_0 + n\nabla y_0 + \frac{n(n+1)}{2!}\nabla^2 y_0 + \ldots$$

$$y(7.5) = ?$$

$$x_0 + nh = 7.5 \quad \Rightarrow \quad 8 + n(1) = 7.5$$

$$n = 7.5 - 8 = -0.5$$

$$n = -0.5$$

$$\therefore y(7.5) = 512 + (-0.5)(169) + \frac{(-0.5)(-0.5+1)}{2!}(42) + \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}$$

$$= 512 - 84.5 + (-0.5)(0.5)(21) + \frac{(-0.5)(0.5)(1.5)(6)}{6}$$

$$= 512 - 84.5 - 5.25 - 0.375$$

$$f(7.5) = 421.875$$

$$\therefore \mathbf{f(7.5) = 421.875}$$

Note : The backward difference can also be defined in the following way, where h being the length of the interval.

$$\nabla f(x) = f(x) - f(x-h)$$
$$\nabla^2 f(x) = \nabla [\nabla f(x)]$$
$$= \nabla [f(x) - f(x-h)]$$
$$= \{f(x) - f(x-h)\} - \{f(x-h) - f(x-2h)\}$$
$$= f(x) - 2f(x-h) + f(x-2h) \text{ and so on.}$$

The differences are shown below

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
$x_{-4} = x_0 - 4h$	y_{-4}				
		∇y_{-3}			
			$\nabla^2 y_{-2}$		
$x_{-3} = x_0 - 3h$	y_{-3}			$\nabla^3 y_{-1}$	
		∇y_{-2}			$\nabla^4 y_0$
			$\nabla^2 y_{-1}$		
$x_{-2} = x_0 - 2h$	y_{-2}			$\nabla^3 y_0$	
		∇y_{-1}			
			$\nabla^2 y_0$		
$x_{-1} = x_0 - h$	y_{-1}				
		∇y_0			
$x_0 = y_0$	y_0				

Newton's Backward Interpolation Formula

We know that $\nabla y_1 = y_1 - y_0$ (or) $(1 - \nabla) y_1 = y_0$ ...(1)

(or) $y_1 = (1 - \nabla)^{-1} y_0$ ...(2)

Also we know that $y_1 = (1 + \Delta) y_0$

[By definition of forward difference operator]

From (1) and (2) we get,

$$(1 - \nabla)^{-1} \equiv (1 + \Delta)$$

Hence $y_n = (1 + \Delta)^n y_0 = (1 - \nabla)^{-n} y_0$

$$= \left[1 + n\nabla + \frac{n(n+1)}{2!}\nabla^2 + \frac{n(n+1)(n+2)}{3!}\nabla^3 + \dots\right] y_0$$

$y(x_0 + nh)$

$$= y_0 + n\nabla y_0 + \frac{n(n+1)}{2!}\nabla^2 y_0 + \frac{n(n+1)(n+2)}{3!}\nabla^3 y_0 + \dots$$

... backward interpolation formula.

16. From the table of values

x	1.46	1.47	1.48	1.49
$f(x)$	0.885604	0.885633	0.885747	0.885945

Find $f(1.4684)$. *[Ans. 0.886226]*

17. A rod is rotating in a plane. The following table gives the angle θ (radians) through which the rod has turned for various values of time t seconds.

t	0	0.2	0.4	0.6	0.8	1.0	1.2
θ	0.	0.12	0.49	1.12	2.02	3.20	4.67

Obtain the value of θ when $t = 0.5$. *[Ans. 0.8]*

18. From the following table calculate the value of y when $x = 218$.

x	100	150	200	250	300	350	400
y	10.63	13.03	15.04	16.81	18.42	19.90	21.27

[Ans. 15.7]

19. Calculate the value of y when $x = 105$ for the following data using Newton's forward formula of interpolation. *[Ans. 8666]*

x	80	85	90	95	100
y	5026	5674	6362	7088	7854

BACKWARD DIFFERENCES

We use another operator called the backward difference operator ∇ and is defined by

$$\nabla y_n = y_n - y_{n-1}$$

For $n = 0, 1, 2, \ldots$ we get

$$\nabla y_0 = y_0 - y_{-1}$$

$$\nabla y_1 = y_1 - y_0$$

$$\nabla y_2 = y_2 - y_1, \text{ and so on.}$$

The second backward difference is

$$\begin{aligned}
\nabla^2 y_n &= \nabla(\nabla y_n) \\
&= \nabla(y_n - y_{n-1}) \\
&= \nabla y_n - \nabla y_{n-1} \\
&= (y_n - y_{n-1}) - (y_{n-1} - y_{n-2}) \\
&= y_n - 2y_{n-1} + y_{n-2}
\end{aligned}$$

Similarly the third backward difference is

$$\begin{aligned}
\nabla^3 y_n &= \nabla^2 y_n - \nabla^2 y_{n-1} \\
&= (y_n - 2y_{n-1} + y_{n-2}) - (y_{n-1} - 2y_{n-2} + y_{n-3}) \\
&= y_n - 3y_{n-1} + 3y_{n-2} - y_{n-3} \text{ and so on.}
\end{aligned}$$

The Newton's forward difference formula is

$$y(x_0 + nh) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \ldots$$

We have to find $y(1.85)$

$$x_0 + nh = 1.85$$

$$1.7 + n \cdot (0.1) = 1.85$$

$$n = \frac{1.85 - 1.7}{0.1} = 1.5$$

Hence $y(1.85) = 5.4739 + (1.5)(0.5757) + \frac{(1.5)(1.5-1)}{2}(0.0606)$

$$+ \frac{(1.5)(1.5-1)(1.5-2)}{6}(0.0063) + \text{negligible terms}$$

$$= 5.4739 + 1.5 \times 0.5757 + 0.375 \times 0.0606 - 0.0625 \times 0.0063$$

$\therefore \quad e^{1.85} = 6.3598$

Example 5 : ***From the following data, find θ at $x = 43$***

X	40	50	60	70	80	90
θ	184	204	226	250	276	304

Solution :

The difference table is as follows.

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
$40 (x_0)$	$184 (y_0)$			
		$20 (\Delta y_0)$		
50	204		$2 (\Delta^2 y_0)$	
		22		$0 (\Delta^3 y_0)$
60	226		2	
		24		0
70	250		2	
		26		0
80	276		2	
		28		
90	304			

Here $x_0 = 40$, $y_0 = 184$, $h = 10$.

The Newton's forward interpolation formula is

$$y(x_0 + nh) = y_0 + n\,\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \ldots$$

$$y(43) = ?$$

$$x_0 + nh = 43$$

$$40 + n(10) = 43$$

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