1

Empirical Laws and Curve Fittings

1.1. Introduction

In practical life and also in Engineering and Science, we come across experiments which involve two quantities. An anxiety arises in us to know the relationship which connects the two quantities. For example, we measure the heights and weights of students in our class and represent those values by x_i and y_i, $i = 1, 2, ..., n$ (assume there are n students). Now, we would like to know whether there is any relation between x and y. This relation, when written as an equation $y = f(x)$, is called an emperical equation. Many times we may not be able to get an exact relation but we may get only an approximate curve. This approximating curve is an empirical equation and the method of finding such an approximating curve is called curve fitting. The constants occuring in the equation of the approximating curve can be found by various methods. If (x_i, y_i), $i = 1, 2, ..., n$ are the n paired data which are plotted on the graph sheet, it is possible to draw a number of smooth curves passing through the points. So, the approximating curve is not unique.

1.2. The linear law

Suppose that the relationship between the variables x and y is linear.

Assume it to be $y = ax + b$ ...(1)

That is, if the points (x_i, y_i) are plotted in the graph-sheet, they should lie on a straight line. 'a' is the slope of the straight and b is the y-intercept. Taking any two points (x_1, y_1) and (x_2, y_2) which are apart, slope of the straight line $a = \dfrac{y_2 - y_1}{x_2 - x_1}$; b can be found by knowing the y-intercept or by substituting any point on the line. Substituting a and b in (1), we get the straight line required.

1.3. Laws reducible to linear law

(1) Law of the pattern $y = ax^n$ where a and n are constants.

Let $y = ax^n$ be the law.

2. The straight line passes through the two points $(\bar{x}_1, \bar{y}_1)$ and $(\bar{x}_2, \bar{y}_2)$. Hence its equation is

$$\frac{y-\bar{y}_1}{x-\bar{x}_1} = \frac{\bar{y}_1-\bar{y}_2}{\bar{x}_1-\bar{x}_2}$$

3. In practice, we shall divide the data into two groups so that both contain the same number of points.

4. The equations got is only an approximate fit to the data given.

Example 2. *Find a straight line fit of the form $y = a + bx$, by the method of group averages for the following data :*

x :	0	5	10	15	20	25
y :	12	15	17	22	24	30

(M.K.U., 1973)

Solution. Let us divide the data into two groups each containing three sets of data.

Group I :	x	y	Group II :	x	y
	0	12		15	22
	5	15		20	24
	10	17		25	30
Sum :	15	44		60	76

$$\therefore \quad \bar{x}_1 = \frac{15}{3} = 5 ; \qquad \bar{y}_1 = \frac{44}{3} = 14{\cdot}6666$$

$$\bar{x}_2 = \frac{60}{3} = 20 ; \qquad \bar{y}_2 = \frac{76}{3} = 25{\cdot}3333$$

Substituting the average values $\bar{x}$ in $\bar{y} = a + b\bar{x}$, we get

$$a + 5b = 14{\cdot}6666 \qquad \ldots(1)$$

$$a + 20b = 25{\cdot}3333 \qquad \ldots(2)$$

(2) – (1) gives, $15b = 10{\cdot}6667$

$$b = 0{\cdot}71111$$

$$\therefore \quad a = 14{\cdot}6666 - 5\,(0{\cdot}71111) = 11{\cdot}11105$$

Hence, the straight is $y = 11{\cdot}11105 + 0{\cdot}71111\,x$

Aliter: The required equation is the straight line joining the points (5, 14·6666) and (20, 25·3333). Its equation is,

$$\frac{y - 14{\cdot}6666}{x-5} = \frac{25{\cdot}3333 - 14{\cdot}6666}{20-5} = \frac{10{\cdot}6667}{15} = 0{\cdot}71111$$

i.e.,

$$y = 14{\cdot}6666 + 0{\cdot}71111\,x - 5 \times (0{\cdot}71111)$$

$$y = 0{\cdot}71111\,x + 11{\cdot}11105$$

Example 3. *The following table gives corresponding values of x and y. Obtain an equation of the form $y = ax + bx^2$.*

x :	1·1	2·0	3·2	4·0	5·5	6·3
y :	5·3	14·2	30·1	43·8	77·3	97·8

Solution. Rewriting the equation as $\frac{y}{x}=a+bx$, put $\frac{y}{x}=Y$.

Hence the equation is $Y=a+bX$

	Group I		Group II	
	$x=X$	$Y=\frac{y}{x}$	$x=X$	$Y=\frac{y}{x}$
	1·1	4·82	4·0	10·95
	2·0	7·10	5·5	14·05
	3·2	9·41	6·3	15·52
Sum :	6·3	21·33	15·8	40·52

$$\bar{X}_1 = \frac{6\cdot3}{3} = 2\cdot1, \qquad \bar{Y}_1 = \frac{21\cdot33}{3} = 7\cdot11$$

$$\bar{X}_2 = \frac{15\cdot8}{3} = 5\cdot2666, \quad \bar{Y}_2 = \frac{40\cdot52}{3} = 13\cdot5066$$

$\therefore$ The line joining $(\bar{X}_1, \bar{Y}_1)$ and $(\bar{X}_2, \bar{Y}_2)$ is

$$\frac{Y-7\cdot11}{X-2\cdot1} = \frac{13\cdot5066-7\cdot11}{5\cdot2666-2\cdot1} = \frac{6\cdot3966}{3\cdot1666} = 2\cdot02002$$

i.e., $$Y = 7\cdot11+2\cdot02X-2\cdot02\times2\cdot1$$
$$Y = 2\cdot02X+2\cdot868$$
$$\frac{y}{x} = 2\cdot02X+2\cdot868$$

i.e., $y = 2\cdot02x^2+2\cdot868x$ is the required equation.

(Here $a=2\cdot868,\ b=2\cdot02$)

Example 4. *The following numbers relate to the flow of water over a triangular notch.*

H :	1·2	1·4	1·6	1·8	2·0	2·4
Q :	4·2	6·1	8·5	11·5	14·9	23·5

If the law is $Q=CH^n$, find C and n by the method of group averages.

Solution. $Q=CH^n$

$$\log_{10}Q = \log_{10}C + n\log_{10}H$$

$y=a+nx$ where $\log_{10}Q=y$, $\log_{10}H=x$ and $a=\log_{10}C$

	Group I		Group II	
	$x=\log_{10}H$,	$y=\log_{10}Q$	$x=\log_{10}H$,	$y=\log_{10}Q$
	0·07918	0·62325	0·25527	1·06070
	0·14163	0·78533	0·30103	1·17319
	0·20412	0·92942	0·38021	1·37107
Sum :	0·42493	2·33800	0·93651	3·60496

$$\bar{x}_1 = \frac{0\cdot42493}{3} = 0\cdot141643; \qquad \bar{y}_1 = \frac{2\cdot3380}{3} = 0\cdot77933$$

$$\bar{x}_2 = \frac{0\cdot93651}{3} = 0\cdot31217; \qquad \bar{y}_2 = \frac{3\cdot60496}{3} = 1\cdot20165$$

We plot the points (x_i, y_i) by P_i, $i = 1, 2, \ldots, n$. The ordinates at $x = x_i$, meets the straight line at Q_i, $i = 1, 2, \ldots, n$.

At $x = x_i$, $P_i M_i = y_i$, and $Q_i M_i = a + bx_i$

$P_i M_i = y_i$ = observed value of y at $x = x_i$

$Q_i M_i = a + bx_i$ = expected value of y at $x = x_i$

$d_i = Q_i P_i = M_i P_i - M_i Q_i = y_i - (a + bx_i)$ which is the difference between the observed value y and the expected value of y is called the residual at $x = x_i$. In general, d_i's may be positive or negative.

Principle: The method of group averages is based on the *principle* (assumption) that the sum of the residuals at all points is *Zero*.

i.e.,
$$\Sigma d_i = 0 \quad \ldots(2)$$

Since we require two equations to find two unknowns a and b, we divide the given data into two groups, first one containing γ sets of values and the other containing $(n - \gamma)$ sets of values. We apply the principle $\Sigma d_i = 0$ for each group. Hence

$$\sum_{i=1}^{\gamma} [y_i - (a + bx_i)] = 0 \quad \text{and} \quad \sum_{i=r+1}^{n} [y_i - (a + bx_i)] = 0$$

$$\therefore \quad \sum_{1}^{\gamma} y_i = \sum_{1}^{\gamma} (a + bx_i) \quad \text{and} \quad \sum_{r+1}^{n} y_i = \sum_{r+1}^{n} (a + bx_i)$$

i.e.,
$$\gamma a + b \sum_{1}^{\gamma} x_i = \sum_{1}^{\gamma} y_i \quad \text{and} \quad (n - \gamma) a + b \sum_{r+1}^{n} x_i = \sum_{r+1}^{n} y_i$$

Hence,
$$a + b \frac{\sum_{1}^{\gamma} x_i}{r} = \frac{\sum_{1}^{\gamma} y_i}{r} \quad \ldots(3)$$

and
$$a + b \frac{\sum_{\gamma+1}^{n} x_i}{(n - r)} = \frac{\sum_{\gamma+1}^{n} y_i}{(n - r)} \quad \ldots(4)$$

That is,
$$a + b\bar{x}_1 = \bar{y}_1 \quad \text{and} \quad a + b\bar{x}_2 = \bar{y}_2 \quad \ldots(5)$$

where $\bar{x}_1, \bar{y}_1$ are the averages of x's and y's of the first group and $\bar{x}_2, \bar{y}_2$ are the averages of x's and y's of the second group.

Equations (5) determine a and b and using these values in (1) we get the relation $y = a + bx$.

Note 1. The answer is not unique since the grouping can be done in many ways. This is an important defect of this method.

The equations are

$$0.77933 = a + n\,(0.14164)$$
$$1.20165 = a + n\,(0.31217)$$
$$n\,(0.31217 - 0.14164) = 1.20165 - 0.77933$$
$$0.17053\,n = 0.42232$$
$$\therefore \quad n = 2.47651$$
$$a = 0.77933 - 2.47651 \times 0.14164 = 0.428557$$

Hence $\quad C = 10^{a} = 10^{0.428557} = 2.68261$

Law is $\quad \mathbf{Q = 2.68261\, H^{2.47651}}$

Example 5. *Using the following table, fit a curve of the form* $y = ax^{b}$. *Find a and b and find the formula by the method of group averages.*

x :	*10*	*20*	*30*	*40*	*50*	*60*	*70*	*80*
y :	*1·06*	*1·33*	*1·52*	*1·68*	*1·81*	*1·91*	*2·01*	*2·11*

(M.K.U.)

Solution. $\log y = \log a + b \log x$

Let $\quad Y = \log_{10} y,\quad A = \log_{10} a,\quad X = \log_{10} x$

We get $\quad Y = A + bx \quad \ldots(1)$

Divide into two groups each containing 4 data.

	Group I X	Group I Y	Group II X	Group II Y
	1·0	0·0253	1·6990	0·2577
	1·3010	0·1239	1·7782	0·2810
	1·4771	0·1818	1·8451	0·3032
	1·6020	0·22531	1·9031	0·3243
Sum :	5·3801	0·5563	7·2254	1·1662

$$\overline{X}_1 = \frac{5.3801}{4} = 1.3450, \quad \overline{Y}_1 = \frac{0.5563}{4} = 0.1391$$

$$\overline{X}_2 = \frac{7.2254}{4} = 1.8064, \quad \overline{Y}_2 = \frac{1.1662}{4} = 0.2916$$

$$\therefore \quad 0.1391 = A + b\,(1.3450)$$
$$0.2916 = A + b\,(1.8064)$$
$$\therefore \quad b\,(1.3450 - 1.8064) = 0.1391 - 0.2916$$
$$b = \frac{0.1525}{0.4614} = 0.3305$$
$$A = 0.1391 - 0.3305 \times 1.3450 = -0.3054$$
$$a = 10^{-0.3054} = 0.4950 \quad \therefore \quad \mathbf{y = 0.495\, x^{0.3305}}$$

Example 6. *Convert the equation* $y = \dfrac{b}{x\,(x-a)}$ *to a linear form and hence determine a and b. Which will best fit the following data :*

$$f(0.71875) = 0.71875 - \cos(0.71875) = ...$$

The root lies between 0.71875 and 0.75

$$x_4 = \frac{1}{2}(0.71875 + 0.75) = 0.73438$$

$$f(0.73438) = -0.0078664 = -ve$$

∴ The root lies between 0.73438 and 0.75

$$x_5 = 0.742190$$

$$f(0.74219) = 0.0051999 = +ve$$

$$x_6 = \frac{1}{2}(0.73438 + 0.742190) = 0.73829$$

$$f(0.73829) = -0.0013305$$

The root lies between 0.73829 and 0.74219

$$x_7 = \frac{1}{2}(0.73829 + 0.74219) = 0.7402$$

$$f(0.7402) = 0.7402 - \cos(0.7402) = 0.0018663$$

The root lies between 0.73829 and 0.7402

$$x_8 = 0.73925$$

$$f(0.73925) = 0.00027593$$

$$x_9 = 0.7388 \qquad \text{(correct to 4 places)}$$

The root is 0.7388.

Example 4. *Find the positive of $x^4 - x^3 - 2x^2 - 6x - 4$ ... bisection method.*

Solution. Let $\quad f(x) = x^4 - x^3 - 2x^2 - 6x - 4$

$f(2)$ and $f(3)$ are opposite in sign since $f(2) = -ve$ and $f(3)$ ...

Therefore the root lies between 2 and 3

$$x_0 = \frac{2+3}{2} = 2.5$$

$f(2.5) = -ve$; hence root lies between 2.5 and 3

∴
$$x_1 = \frac{2.5+3}{2} = 2.75$$

Proceeding in the same manner, the sequence of mid-points ... 2.75, 2.63, 2.69, 2.72, 2.735, 2.728, 2.7315, 2.7298, 2.7307, ... 2.7313, 2.7314, 2.7315,...

$$f(2.7315) = -0.0232$$

The root of the equation is 2.7315 approximately.

Example 5. *Using bisection method, find the negative ... $x^3 - 4x + 9 = 0$ by bisection method.*

Solution. Let

$$f(x) = x^3 - 4x + 9$$

$$f(-x) = -x^3 + 4x + 9$$

The negative root of $f(x) = 0$ is the positive root of $f(-x) = ...$

$\therefore$ We will find the positive root of $f(-x)=0$, firstly.

i.e., $\phi(x)=x^3-4x-9=0$

$\phi(2)=-ve$ and $\phi(3)=+ve$

$\therefore$ The root lies between 2 and 3.

Hence $x_0=\frac{2+3}{2}=2\cdot5$

$$\phi(2\cdot5)=(2\cdot5)^3-4(2\cdot5)-9=-ve$$

Therefore, the root lies between 2·5 and 3.

Hence, $x_1=\frac{1}{2}(2\cdot5+3)=2\cdot75$

$$\phi(2\cdot75)=+ve$$

$\therefore$ The root lies between 2·5 and 2·75

$$x_2=\tfrac{1}{2}(2\cdot5+2\cdot75)=2\cdot625$$

$$\phi(2\cdot625)=(2\cdot625)^3-4(2\cdot625)-9=-1\cdot4121=-ve$$

The root lies between 2·625 and 2·75.

$$x_3=\tfrac{1}{2}(2\cdot625+2\cdot75)=2\cdot6875$$

$$\phi(2\cdot6875)=-ve$$

$\therefore$ The root lies between 2·6875 and 2·75.

$$x_4=\tfrac{1}{2}(2\cdot6875+2\cdot75)=2\cdot71875$$

$$\phi(2\cdot71875)=+ve$$

$\therefore$ The root lies between 2·6875 and 2·71875.

$\therefore$ $x_5=\frac{1}{2}(2\cdot6875+2\cdot71875)=2\cdot703125$

$$\phi(2\cdot703125)=(2\cdot703125)^3-4(2\cdot703125)-9=-ve$$

$\therefore$ The root lies between 2·703125 and 2·71875.

$$x_6=\tfrac{1}{2}(2\cdot703125+2\cdot71875)=2\cdot710938$$

Proceeding in the same way,

$$x_7=2\cdot707031,\ x_8=2\cdot705078,\ x_9=2\cdot706054,$$

$$x_{10}=2\cdot70654,\ x_{11}=2\cdot706297,\ x_{12}=2\cdot706418,$$

$$x_{13}=2\cdot70648,\ x_{14}=2\cdot70651 \text{ etc.}$$

We can conclude the root to be 2·7065 for $\phi(x)=0$

Hence the negative root of the given equation is $-2\cdot7065$.

3·2. Iteration method (or Method of successive approximations)

Suppose we want the approximate roots of the equation

$$f(x)=0 \quad ...(1)$$

Now, write the equation (1) in the form

$$x=\phi(x) \quad ...(2)$$

$\therefore$ The root lies between 2·9375 and 2·9532

$$x_7 = \frac{1}{2}(2\cdot9375 + 2\cdot9532) = 2\cdot9454$$

$$f(2\cdot9454) = +ve$$

The root lies between 2·9375 and 2·9454

$$x_8 = \frac{1}{2}(2\cdot9375 + 2\cdot9454) = 2\cdot9415$$

$$f(2\cdot9415) = -ve$$

The root lies between 2·9415 and 2·9454.

$$x_9 = \frac{1}{2}(2\cdot9415 + 2\cdot9454) = 2\cdot9435$$

$$f(2\cdot9435) = +ve$$

The root lies between 2·9415 and 2·9435.

$$x_{10} = 2\cdot9425$$

$$f(2\cdot9425) = -ve$$

The root lies between 2·9425 and 2·9435.

$$x_{11} = 2\cdot9430$$

$$f(2\cdot9430) = +ve$$

$$x_{12} = 2\cdot94275$$

$$x_{13} = 2\cdot942875$$

Approximate root is 2·9429.

Example 3. *Find the positive root of $x - \cos x = 0$ by bisection method.*

Solution. Let $f(x) = x - \cos x$

$$f(0) = -ve,\ f(0\cdot5) = 0\cdot5 - \cos(0\cdot5) = -0\cdot37758$$

$$f(1) = 1 - \cos 1 = 0\cdot45970$$

Hence, the root lies between 0·5 and 1.

$$x_0 = \frac{0\cdot5 + 1}{2} = 0\cdot75$$

$$f(0\cdot75) = 0\cdot75 - \cos(0\cdot75) = 0\cdot018311 = +ve$$

$\therefore$ The root lies between 0·5 and 0·75.

$$x_1 = \frac{0\cdot5 + 0\cdot75}{2} = 0\cdot625$$

$$f(0\cdot625) = 0\cdot625 - \cos(0\cdot625) = -0\cdot18596$$

The root lies between 0·625 and 0·750.

$$x_2 = \frac{1}{2}(0\cdot625 + 0\cdot750) = 0\cdot6875$$

$$f(0\cdot6875) = -0\cdot085335$$

$\therefore$ The root lies between 0·6875 and 0·75.

$$x_3 = \frac{1}{2}(0\cdot6875 + 0\cdot75) = 0\cdot71875$$

$$x_{10} = \frac{1}{2}(1{\cdot}3243 + 1{\cdot}3248) = 1{\cdot}32455$$

$$f(1{\cdot}32455) = -ve$$

The root lies between 1·3248 and 1·32455

$$\therefore \quad x_{11} = \frac{1}{2}(1{\cdot}3248 + 1{\cdot}32455) = 1{\cdot}3247$$

$$f(1{\cdot}3247) = -ve$$

$\therefore$ The root lies between 1·3247 and 1·3248

Hence, $$x_{12} = \frac{1}{2}(1{\cdot}3247 + 1{\cdot}3248) = 1{\cdot}32475$$

Therefore, the approximate root is 1·32475

(This is not correct to 5 decimal places).

Example 2. *Assuming that a root of $x^3 - 9x + 1 = 0$ ... interval (2, 4), find that root by bisection method.*

Solution. Let $f(x) = x^3 - 9x + 1$

$f(2) = -ve$ and $f(4) = +ve$

Therefore, a root lies between 2 and 4

Let $$x_0 = \frac{2+4}{2} = 3$$

Now $f(3) = +ve$; hence the root lies between 2 and 3

$$x_1 = \frac{2+3}{2} = 2{\cdot}5$$

$$f(x_1) = f(2{\cdot}5) = -ve$$

The root lies between 2·5 and 3.

$$x_2 = \frac{2{\cdot}5+3}{2} = 2{\cdot}75$$

$$f(2{\cdot}75) = -ve$$

The root lies between 2·75 and 3

$$x_3 = \frac{1}{2}(2{\cdot}75 + 3) = 2{\cdot}875$$

$$f(x_3) = f(2{\cdot}875) = -ve$$

Therefore, the root lies between 2·875 and 3

$$x_4 = \frac{1}{2}(2{\cdot}875 + 3) = 2{\cdot}9375$$

$$f(2{\cdot}9375) = -ve$$

$\therefore$ The root lies between 2·9375 and 3

$$x_5 = \frac{1}{2}(2{\cdot}9375 + 3) = 2{\cdot}9688$$

$$f(2{\cdot}9688) = +ve$$

The root lies between 2·9688 and 2·9375

$$x_6 = \frac{1}{2}(2{\cdot}9375 + 2{\cdot}9688) = 2{\cdot}9532$$

$f(2{\cdot}9532)$ ...

$$= \xi_n - \frac{f(\alpha) + \xi_n f'(\alpha) + \frac{\xi_n^2 f''(\alpha)}{2!} + \dots}{f'(\alpha) + \xi_n f''(\alpha) + \dots}$$

[By Taylor's expansion]

$$= \xi_n - \frac{f(\alpha) + \xi_n f'(\alpha) + \frac{\xi_n^2 f''(\alpha)}{2}}{f'(\alpha) + \xi_n f''(\alpha)}$$

[Omitting derivatives of order higher than two]

$$= \frac{\xi_n f'(\alpha) + \xi_n^2 f''(\alpha) - f(\alpha) - \xi_n f'(\alpha) - \frac{\xi_n^2 f''(\alpha)}{2}}{f'(\alpha) + \xi_n f''(\alpha)}$$

$$= \frac{\frac{1}{2}\xi_n^2 f''(\alpha)}{f'(\alpha) + \xi_n f''(\alpha)}$$ [$f(\alpha) = 0$ since α is a root of $f(x)=0$]

$$= \frac{\xi_n^2 f''(\alpha)}{2 f'(\alpha)\left[1 + \frac{\xi_n f''(\alpha)}{f'(\alpha)}\right]}$$

$$= \xi_n^2 \cdot \frac{f''(\alpha)}{2f'(\alpha)}$$

[$\because \frac{\xi_n f''(\alpha)}{f'(\alpha)}$ is a very small quantity and therefore we neglect it]

$$= k \cdot \xi_n^2, \text{ where } k = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}$$

which shows that in each step, the subsequent error is proportional to the square of the previous error and as such the convergence is quadratic or its order of convergence is two.

❑ Condition for convergence of Newton - Raphson method

Newton's formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= \phi(x_n)$$

$$x_{n+1} = \phi(x_n)$$

which is similar to the iteration method.

Hence in analogy to the convergence of iterative method, Newton - Raphson method converges when,

$$|\phi'(x)| < 1.$$

Put $n = 0$ in (A), we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)}$$

$$= 2 - \left[\frac{(2)^4 - 2 - 10}{4(2)^3 - 1}\right] = 2 - \left[\frac{4}{4 \times 8 - 1}\right]$$

$$= 2 - \frac{4}{31} = 2 - 0 \cdot 1290$$

$$x_1 = 1.871$$

Put $n = 1$ in (A), we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1 \cdot 871 - \frac{f(1.871)}{f'(1.871)}$$

$$= 1 \cdot 871 - \left[\frac{(1.871)^4 - 1.871 - 10}{4(1.871)^3 - 1}\right]$$

$$= 1 \cdot 871 - \frac{0.3835}{25.199}$$

$$= 1 \cdot 871 - 0 \cdot 0152$$

$$= 1 \cdot 8557$$

$$x_2 = 1.856 \qquad \text{(correct 3 decimal places)}$$

Put $n = 2$ in (A) we get,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 1.856 - \frac{f(1.856)}{f'(1.856)}$$

$$= 1.856 - \left[\frac{(1.856)^4 - (1.856) - 10}{4(1.856)^3 - 1}\right]$$

$$= 1 \cdot 856 - \frac{0.010}{24.574}$$

$$= 1 \cdot 856 - 0 \cdot 000406$$

$$= 1 \cdot 8555$$

$$x_3 = 1.856 \qquad \text{(correct to 3 decimal places)}$$

Here $x_2 = x_3 = 1.856$

$\therefore$ **The root is $1 \cdot 856$**

Table

Iteration	x_r	$f(x_r)$
1	1.871	0.3835
2	1.856	0.010
3	1.856	0.010

Put $n = 1$ in (A), we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0\cdot67 - \left[\frac{(0.67)^3 - 6\,(0.67) + 4}{3\,(0.67)^2 - 6}\right]$$

$$= 0\cdot67 - \frac{0.28}{3\,(0.449) - 6}$$

$$= 0\cdot67 - \frac{0.28}{1.347 - 6} = 0\cdot67 - \frac{0.28}{-4.653}$$

$$= 0\cdot67 + 0\cdot060$$

$$x_2 = 0.73$$

Put $n = 2$ in (A) we get,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0\cdot73 - \frac{f(0.73)}{f'(0.73)}$$

$$= 0\cdot73 - \left[\frac{(0.73)^3 - 6\,(0.73) + 4}{3\,(0.73)^2 - 6}\right]$$

$$= 0\cdot73 - \frac{0.009}{3\,(0.533) - 6} = 0\cdot73 - \frac{0.009}{-4.401}$$

$$= 0\cdot73 + 0\cdot00204 = 0\cdot732$$

$$x_3 = 0.73 \qquad \text{(correct two decimal pla}$$

Here $x_2 = x_3 = 0.73$

∴ The root is $0\cdot73$

Table

Iteration	x_r	$f(x_r)$
1	0.67	0.28
2	0.73	0.009
3	0.73	0.009

Example 3 : *Find the positive root of $x^4 = x + 10$ correct to three deci places using Newton's Raphson method.*

Solution :

Let $f(x) = x^4 - x - 10$; $f'(x) = 4x^3 - 1$

Now $f(1) = (1)^4 - 1 - 10$

$= 1 - 1 - 10 = -10$ (– ve)

$f(2) = (2)^4 - 2 - 10$

$= 16 - 2 - 10 = 4$ (+ ve)

Since $|f(2)| < |f(1)|$, a root is nearer to 2.

Take $x_0 = 2$

Newton's formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{...(A)}$$

Scanned by CamScanner

Putting $n = 2$ in (1), we get the **third approximation** x_3 to the root, given by

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.2586 - \frac{(0.2586)^3 - 2(0.2586) + 0.5}{3(0.2586)^2 - 2}$$

$$x_3 = \mathbf{0.2586}$$

Hence the smallest positive root is 0.2586.

Table

Iteration	x_r	$f(x_r)$
1	0.25	0.0156
2	0.2586	0.0000936
3	0.2586	0.0000936

Example 2 : *Using Newton's iterative method find the root between 0 and 1 of $x^3 = 6x - 4$ correct to two decimal places.* *[A.U. May 2000]*

Solution :

Let $f(x) = x^3 - 6x + 4$; $f'(x) = 3x^2 - 6$

Now $f(0) = 0 - 0 + 4 = 4 = +\text{ve}$

$f(1) = (1)^3 - 6(1) + 4$

$= 1 - 6 + 4 = -1 = -\text{ve}$

Hence the roots lies between 0 and 1.

Here $|f(0)| = 4$

$|f(1)| = 1$

Since $|f(1)| < |f(0)|$, the required root is nearer to 1.

Newton's formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{... (A)}$$

Take $x_0 = 1$.

Put $n = 0$ in (A), we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)}$$

$$= 1 - \frac{[(1)^3 - 6(1) + 4]}{3(1)^2 - 6}$$

$$= 1 - \frac{(-1)}{-3} = 1 - \frac{1}{3} = \frac{2}{3}$$

$$= 0.666 \qquad \text{[correct to two decimal places]}$$

$$x_1 = \mathbf{0.67}$$

But $\quad \phi'(x) = \frac{d}{dx}\left[x - \frac{f(x)}{f'(x)}\right]$

$$= 1 - \left\{\frac{f'(x)\cdot f'(x) - f(x)\cdot f''(x)}{[f'(x)]^2}\right\}$$

$$= \frac{f(x)\cdot f''(x)}{[f'(x)]^2}$$

Hence the condition for convergence of Newton-Raphson metho

$$\left|\frac{f(x)\cdot f''(x)}{[f'(x)]^2}\right| < 1$$

$$|f(x)\cdot f''(x)| < |f'(x)|^2$$

This is the condition for convergence of Newton's-Rap method.

Example 1: Find the smallest positive root of the equation $x^3 - 2x + 0.5 = 0$.

Solution :

Given $\quad f(x) = x^3 - 2x + 0.5, \quad f'(x) = 3x^2 - 2$

Now $\quad f(0) = 0.5$ (+ve)

$f(1) = -0.5$ (–ve)

Hence the root lies between 0 and 1. Since the value of $f(x)$ at x = very close to zero than the value of $f(x)$ at $x = 1$, we can say that the is very close to 0. Therefore we can assume that $x_0 = 0$, is the approximation to the root.

Newton's-Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Putting $n = 0$ in (1), we get the **first approximation** x_1 to the given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{0.5}{-2}$$

$$x_1 = \mathbf{0.25}$$

Putting $n = 1$ in (1), we get the **second approximation** x_2 to the given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.25 - \frac{(0.25)^3 - 2(0.25) + 0.5}{3(0.25)^2 - 2}$$

$$= 0.25 - \frac{0.0156}{-1.8125} = 0.2586$$

$$x_2 = \mathbf{0.2586}$$

$$x_1 - x_0 = \frac{-f(x_0)}{f'(x_0)}$$

(or) $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Similarly, $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In general, $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Thus if we start with a given value x_0, we can obtain $x_1, x_2, x_3, x_4, \ldots$ If any two consecutive values are equal and equal to s then s is one of the roots of $f(x) = 0$.

NOTE:

1. When $f'(x)$ is very large i.e., when the slope is very large the correct value of the root can be found with great rapidity and very little labor. In this case the graph of $f(x) = 0$ is nearly vertical where it crosses the X-axis.

2. The process will evidently fail if $f'(x) = 0$ in the neighbourhood of the root. In such cases the Regula falsi method should be used.

3. If we choose the initial approximation x_0 close to the root then we get the root of the equation very quickly.

4. If the initial approximation, to the root is not given then we can find any two values of x say a and b such that $f(a)$ and $f(b)$ are of opposite signs. If $|f(a)| < |f(b)|$ then 'a' is taken as the first approximation to the root.

5. Newton's–Raphson method is also referred to as the method of tangents.

❑ Rate of convergence of Newton's method

Let α be the exact value of the root of the equation $f(x) = 0$ and ξ_n be a small quantity by which x_n differs from α.

$$x_n = \alpha + \xi_n \quad \ldots(1)$$

so that $$x_{n+1} = \alpha + \xi_{n+1} \quad \ldots(2)$$

By Newton's formula we have,

$$\therefore x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \ldots(3)$$

Using (1) and (2) in (3), we get,

$$\therefore \alpha + \xi_{n+1} = \alpha + \xi_n - \frac{f(\alpha + \xi_n)}{f'(\alpha + \xi_n)}$$

$$\xi_{n+1} = \xi_n - \frac{f(\alpha + \xi_n)}{f'(\alpha + \xi_n)}$$

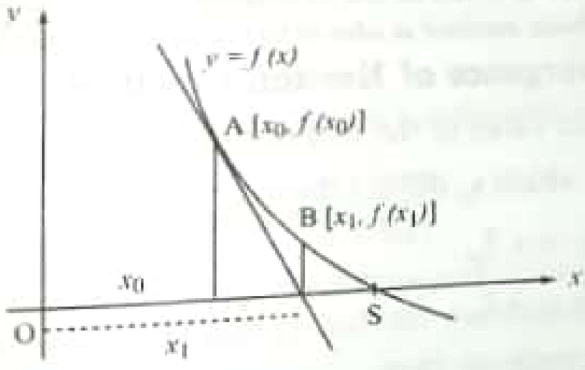
CHAPTER 3

SOLUTION OF EQUATIONS AND EIGENVALUE PROBLEMS

Newton's Method (or) Newton's – Raphson Method

In 17th century Newton discovered a method for estimating a solution or root of an algebraic equation by defining a sequence of numbers that become successively closer and closer to the root sought. His method is best illustrated graphically.

Let s be the actual root of the equation $f(x) = 0$. Draw the graph of the curve $y = f(x)$ as shown in fig. Let a number x_0 be chosen arbitrarily. We then locate the point $(x_0, f(x_0))$ on the curve and draw the tangent line to the curve at that point meeting the x-axis at x_1. x_1 is the first approximation to s. We locate the point $(x_1, f(x_1))$ on the curve and draw the tangent line to the curve at that point. Let this tangent line meet the x-axis at x_2. x_2 is the second approximation to s. We then repeat the process to arrive as closely as possible to s. Now we will derive the formula to arrive at $x_0, x_1, x_2, \ldots$, etc.



The slope of the tangent line at the point $(x_0, f(x_0))$ is $f'(x_0)$. $(x_1, 0)$ and $(x_0, f(x_0))$ are the two points on this tangent line.

$\therefore$ Slope of the tangent line is $\dfrac{0 - f(x_0)}{x_1 - x_0}$

[$\because$ the slope of the line joining the two points (x_1, y_1) and (x_2, y_2) is $\dfrac{y_1 - y_2}{x_1 - x_2}$]

$$f'(x_0) = \frac{-f(x_0)}{x_1 - x_0}$$

Solution. Let $f(x) = x^3 - 2x - 5 = 0$

$$f(2) = -1 = -ve\ ;\ f(3) = 16 = +ve$$

The root lies between 2 and 3 and closer to 2. $f(x) = 0$ can be written as

$$x^3 = 2x + 5\ ;\quad i.e.,\quad x = (2x+5)^{\frac{1}{3}} = \phi(x)$$

$$\phi'(x) = \frac{2}{3} \cdot \frac{1}{(2x+5)^{2/3}}$$

$|\phi'(x)| < 1$ for all x in (2, 3).

Take $x_0 = 2{\cdot}0$

$$x_1 = (2x_0 + 5)^{1/3} = 9^{1/3} = 2{\cdot}0801$$

$$x_2 = (9{\cdot}1602)^{1/3} = 2{\cdot}0924\ ;\ x_3 = (9{\cdot}1848)^{1/3} = 2{\cdot}0942$$

$$x_4 = (9{\cdot}1884)^{1/3} = 2{\cdot}0945\ ;\ x_5 = (9{\cdot}1890)^{1/3} = 2{\cdot}0945$$

Therefore the root is 2·0945.

Example 6. *Find a positive root of $3x - \sqrt{1 + \sin x} = 0$ by iteration method.*

Solution. Writing the given equation as

$$x = \frac{1}{3}\sqrt{1 + \sin x} = \phi(x).\ \ \phi'(x) = \frac{\cos x}{6\sqrt{1 + \sin x}}$$

The root of given equation lies in (0, 1)

since $f(0) = -ve$ and $f(1) = +ve$

In (0, 1), $|\phi'(x)| < 1$ for all x

So, we can use iteration method.

Taking $x_0 = 0{\cdot}4$, $x_1 = \frac{1}{3}\sqrt{1 + \sin(0{\cdot}4)} = 0{\cdot}39291$

$$x_2 = \frac{1}{3}\sqrt{1 + \sin(0{\cdot}39291)} = 0{\cdot}39199$$

$$x_3 = \frac{1}{3}\sqrt{1 + \sin(0{\cdot}39199)} = 0{\cdot}39187$$

$$x_4 = 0{\cdot}39185$$

$$x_5 = 0{\cdot}39185$$

The root is 0·39185.

3·3. Regula Falsi method (or the method of false position)

Consider the equation $f(x) = 0$ and let $f(a)$ and $f(b)$ be of opposite signs. Also, let $a < b$. The curve $y = f(x)$ will meet the x-axis at some point between $A\,(a, f(a))$ and $B\,(b, f(b))$. The equation of the chord joining the two points $A\,(a, f(a))$ and $B\,(b, f(b))$ is $\dfrac{y - f(a)}{x - a} = \dfrac{f(a) - f(b)}{a - b}$. The

Hence, the root lies between 1 and 1·5 ...(1)

Let $$x_0 = \frac{1 + 1\cdot5}{2} = 1\cdot2500$$

$$f(x_0) = f(1\cdot25) = -0\cdot29688$$

Hence the root lies between 1·25 and 1·5 ...(2)

Now, $$x_1 = \frac{1\cdot25 + 1\cdot5}{2} = 1\cdot3750$$

$$f(1\cdot3750) = 0\cdot22461 = +ve$$

The root lies between 1·2500 and 1·3750.

Now $$x_2 = \frac{1\cdot2500 + 1\cdot3750}{2} = 1\cdot3125$$

$$f(1\cdot3125) = -0\cdot051514$$

Therefore, root lies between 1·3750 and 1·3125

Now $$x_3 = \frac{1\cdot3125 + 1\cdot3750}{2} = 1\cdot3438$$

$$f(x_3) = f(1\cdot3438) = 0\cdot082832 = +ve$$

The root lies between 1·3125 and 1·3438

Hence $$x_4 = \frac{1\cdot3125 + 1\cdot3438}{2} = 1\cdot3282$$

$$f(1\cdot3282) = 0\cdot014898$$

Therefore the root lies between 1·3125 and 1·3282

$$x_5 = \frac{1}{2}(1\cdot3125 + 1\cdot3282) = 1\cdot3204$$

$$f(1\cdot3204) = -0\cdot018340$$

The root lies between 1·3204 and 1·3282

$$x_6 = \frac{1}{2}(1\cdot3204 + 1\cdot3282) = 1\cdot3243$$

$$f(1\cdot3243) = -ve$$

Hence, the root lies between 1·3243 and 1·3282

$\therefore$ $$x_7 = \frac{1}{2}(1\cdot3243 + 1\cdot3282) = 1\cdot3263$$

$$f(1\cdot3263) = +ve$$

$\therefore$ The root lies between 1·3243 and 1·3263

$$x_8 = \frac{1}{2}(1\cdot3243 + 1\cdot3263) = 1\cdot3253$$

$$f(1\cdot3253) = +ve$$

The root lies between 1·3243 and 1·3253

$\therefore$ $$x_9 = \frac{1}{2}(1\cdot3243 + 1\cdot3253) = 1\cdot3248$$

$$f(1\cdot3248) = +ve$$

Step 5: Delete the first row and first column and perform steps 2 to 4 on the resulting matrix.

$$\begin{pmatrix} 1 & \frac{1}{7} & \frac{-1}{28} & \frac{8}{7} \\ 0 & \frac{20}{7} & \frac{281}{28} & \frac{160}{7} \\ 0 & \frac{117}{7} & \frac{57}{14} & \frac{229}{7} \end{pmatrix} \leftarrow \text{New Pivot}$$

$$\begin{pmatrix} 1 & \frac{1}{7} & \frac{-1}{28} & \frac{8}{7} \\ 0 & \frac{117}{7} & \frac{57}{14} & \frac{229}{7} \\ 0 & \frac{20}{7} & \frac{281}{28} & \frac{160}{7} \end{pmatrix}$$

R_3 and R_2 are interchanged to move the pivot to the top of new submatrix

$$\begin{pmatrix} 1 & \frac{1}{7} & \frac{-1}{28} & \frac{8}{7} \\ 0 & 1 & \frac{19}{78} & \frac{229}{117} \\ 0 & \frac{20}{7} & \frac{281}{28} & \frac{160}{7} \end{pmatrix} \qquad R_2 \to R_2 \times \frac{7}{117}$$

$$\begin{pmatrix} 1 & \frac{1}{7} & \frac{-1}{28} & \frac{8}{7} \\ 0 & 1 & \frac{19}{78} & \frac{229}{117} \\ 0 & 0 & \frac{1457}{156} & \frac{2020}{117} \end{pmatrix} \qquad R_3 \to R_3 - \frac{20}{7} R_2$$

Step 6: Delete first two rows and two columns. Perform step 5 on the resulting matrix.

$$\begin{pmatrix} 1 & \frac{1}{7} & \frac{-1}{28} & \frac{8}{7} \\ 0 & 1 & \frac{19}{78} & \frac{229}{117} \\ 0 & 0 & \frac{1457}{156} & \frac{2020}{117} \end{pmatrix} \leftarrow \text{New Pivot}$$

$$\begin{pmatrix} 1 & \frac{1}{7} & \frac{-1}{28} & \frac{8}{7} \\ 0 & 1 & \frac{19}{78} & \frac{229}{117} \\ 0 & 0 & 1 & 1.8485 \end{pmatrix} \qquad R_3 \to R_3 \times \frac{156}{1457}$$

Step 7: Use back - substitution to find the solution to the system.

$$z = 1.8485$$

$$y + \frac{19}{78} z = \frac{229}{117}$$

$$y = \frac{229}{117} - \frac{19}{78} \times 1.8485 = 1.50697$$

$$x + \frac{1}{7} y - \frac{1}{28} z = \frac{8}{7}$$

Hence the solution is

$$x_1 = -2,\ x_2 = 3,\ x_3 = 6.$$

Checking :

$$x_1 - x_2 + x_3 = 1$$

$$-2 - 3 + 6 = 1$$

Note 1 : This method fails if the element in the top of the first column is zero. Therefore in this case we can interchange the rows so as to get the non-zero element in the top of the first column.

Note 2 : If we are not interested in the elimination of x, y, z in a particular order, then we can choose at each stage the numerically largest coefficient of the entire coefficient matrix. This requires an interchange of equations and also an interchange of the position of the variables.

Example 2: *Solve the system of equations*

$$28x + 4y - z = 32$$

$$x + 3y + 10z = 24$$

$$2x + 17y + 4z = 35$$

using Gauss elimination method. [Nov. 90 Civil, Apr. 92 ECE]

Solution :

Step 1 : Write the given system in augmented matrix form.

$$\begin{pmatrix} 28 & 4 & -1 & 32 \\ 1 & 3 & 10 & 24 \\ 2 & 17 & 4 & 35 \end{pmatrix}$$

Step 2 : From the first column with non-zero components (called the pivot column) select the component with the largest absolute value. This component is called the pivot.

$$\textit{Pivot} \rightarrow \begin{pmatrix} \mathbf{28} & 4 & -1 & 32 \\ 1 & 3 & 10 & 24 \\ 2 & 17 & 4 & 35 \end{pmatrix}$$

Step 3 : Make the pivot as 1 by dividing the first row by the pivot.

$$\begin{pmatrix} 1 & \frac{1}{7} & \frac{-1}{28} & \frac{8}{7} \\ 1 & 3 & 10 & 24 \\ 2 & 17 & 4 & 35 \end{pmatrix} \qquad R_1 \rightarrow R_1 \div 28$$

Step 4 : Add multiples of the first row to the other rows to make all the other components in the pivot column equal to zero.

$$\begin{pmatrix} 1 & \frac{1}{7} & \frac{-1}{28} & \frac{8}{7} \\ 0 & \frac{20}{7} & \frac{281}{28} & \frac{160}{7} \\ 0 & \frac{117}{7} & \frac{57}{14} & \frac{229}{7} \end{pmatrix} \qquad \begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - 2R_1 \end{array}$$

Step 6: $R_3 \to R_3 \div \frac{189}{26}$

$$\begin{pmatrix} 1 & \frac{-1}{5} & \frac{3}{10} & \frac{23}{10} \\ 0 & 1 & \frac{-7}{13} & \frac{-47}{13} \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

Step 7: Use back substitution to find the solution to the system.

Here $\quad z = 3$

$$y - \frac{7}{13}(z) = \frac{-47}{13}$$

$$13y - 7z = -47$$

$$13y - 7(3) = -47$$

$$13y = -47 + 21$$

$$13y = -26$$

$$y = -2$$

$$x - \frac{1}{5}y + \frac{3}{10}z = \frac{23}{10}$$

$$10x - 2y + 3z = 23$$

$$10x - 2(2) + 3(3) = 23$$

$$10x + 4 + 9 = 23$$

$$10x = 23 - 13$$

$$10x = 10$$

$$x = 1$$

$\therefore \quad x = 1 \;,\; y = -2 \;,\; z = 3$

<u>**Checking :**</u>

$$2x + 10y - 5z = -33$$

$$2(1) + 10(-2) - 5(3) = -33$$

$$2 - 20 - 15 = -33$$

$$-33 = -33$$

Example 4: ***Gauss – elimination method.*** *[A.U. Apr. '92]*

$$3x + y - z = 3$$

$$2x - 8y + z = -5$$

$$x - 2y + 9z = 8$$

Solution :

Step 1: Write the given system of equations in augmented matrix form.

$$\begin{bmatrix} 3 & 1 & -1 & 3 \\ 2 & -8 & 1 & -5 \\ 1 & -2 & 9 & 8 \end{bmatrix}$$

Putting $x = 2{\cdot}001$, $y = 1{\cdot}999$ in (6) we get

$$z = \frac{1}{8}[20 - 2{\cdot}001 - 1{\cdot}999] = 2$$

∴ In the fifth iteration we get

$$x = 2{\cdot}001, \quad y = 1{\cdot}999, \quad z = 2$$

Sixth Iteration

Putting $y = 1{\cdot}999$, $z = 2$ in (4) we get

$$x = \frac{1}{4}[14 - 2(1{\cdot}999) - 2] = 2{\cdot}001$$

Putting $x = 2{\cdot}001$, $z = 2$ in (5) we get

$$y = \frac{1}{5}[10 - 2{\cdot}001 + 2] = 1{\cdot}999$$

Putting $x = 2{\cdot}001$, $y = 1{\cdot}999$ in (6) we get

$$z = \frac{1}{8}[20 - 2{\cdot}001 - 1{\cdot}999] = 2$$

∴ In the sixth iteration we get

$$x = 2{\cdot}001, \quad y = 1{\cdot}999, \quad z = 2$$

Checking :

$$x + y + 8z = 20$$
$$2{\cdot}001 + 1{\cdot}999 + 8(2) = 20$$

Example 3 : *Solve the following system of equations*

$$3x - y + z = 1,$$
$$3x + 6y + 2z = 0,$$
$$3x + 3y + 7z = 4$$

using Gauss – Seidel iteration method.

Solution :

The given system is

$$3x - y + z = 1$$
$$3x + 6y + 2z = 0$$
$$3x + 3y + 7z = 4$$

Clearly the coefficient matrix $\begin{pmatrix} 3 & -1 & 1 \\ 3 & 6 & 2 \\ 3 & 3 & 7 \end{pmatrix}$ is diagonally domi...

Hence we can apply Gauss – Seidel method without any difficulty.

From (1), (2) and (3) we get

$$x = \frac{1}{3}(1 + y - z) \quad \ldots(4)$$

$$y = \frac{1}{6}(0 - 3x - 2z) \quad \ldots(5)$$

$$z = \frac{1}{7}(4 - 3x - 3y) \quad \ldots(6)$$

Putting $x = 2{\cdot}375,\ z = 1{\cdot}9$ in (5) we get

$$y = \frac{1}{5}[10 - 2{\cdot}375 + 1{\cdot}9] = 1{\cdot}905$$

Putting $x = 2{\cdot}375,\ y = 1{\cdot}905$ in (6) we get

$$z = \frac{1}{8}[20 - 2{\cdot}375 - 1{\cdot}905] = 1{\cdot}965$$

$\therefore$ In the second iteration we get

$$\mathbf{x = 2{\cdot}375,\ y = 1{\cdot}905,\ z = 1{\cdot}965}$$

Third Iteration

Putting $y = 1{\cdot}905,\ z = 1{\cdot}965$ in (4) we get

$$x = \frac{1}{4}[14 - 2(1{\cdot}905) - 1{\cdot}965] = 2{\cdot}056$$

Putting $x = 2{\cdot}056,\ z = 1{\cdot}965$ in (5) we get

$$y = \frac{1}{5}[10 - 2{\cdot}056 + 1{\cdot}965] = 1{\cdot}982$$

Putting $x = 2{\cdot}056,\ y = 1{\cdot}982$ in (6) we get

$$z = \frac{1}{8}[20 - 2{\cdot}056 - 1{\cdot}982] = 1{\cdot}995$$

$\therefore$ In the third iteration we get

$$\mathbf{x = 2{\cdot}056,\ y = 1{\cdot}982,\ z = 1{\cdot}995}$$

Fourth Iteration

Putting $y = 1{\cdot}982,\ z = 1{\cdot}995$ in (4) we get

$$x = \frac{1}{4}[14 - 2(1{\cdot}982) - 1{\cdot}995] = 2{\cdot}010$$

Putting $x = 2{\cdot}010,\ z = 1{\cdot}995$ in (5) we get

$$y = \frac{1}{5}[10 - 2{\cdot}010 + 1{\cdot}995] = 1{\cdot}997$$

Putting $x = 2{\cdot}010,\ y = 1{\cdot}997$ in (6) we get

$$z = \frac{1}{8}[20 - 2{\cdot}010 - 1{\cdot}997] = 1{\cdot}999$$

$\therefore$ In the fourth iteration we get

$$\mathbf{x = 2{\cdot}010,\ y = 1{\cdot}997,\ z = 1{\cdot}999}$$

Fifth Iteration

Putting $y = 1{\cdot}997,\ z = 1{\cdot}999$ in (4) we get

$$x = \frac{1}{4}[14 - 2(1{\cdot}997) - 1{\cdot}999] = 2{\cdot}001$$

Putting $x = 2{\cdot}001,\ z = 1{\cdot}999$ in (5) we get

$$y = \frac{1}{5}[10 - 2{\cdot}001 + 1{\cdot}999] = 1{\cdot}999$$

Putting $x = -1{\cdot}552$ in (4) we get

$$y = \frac{1}{25}[15 - 2(-1{\cdot}552)]$$

$$= 0{\cdot}724$$

∴ After sixth iteration we get

$$x = -1{\cdot}552 \;,\; y = 0{\cdot}724$$

Checking :

$$x - 2y = -3$$

$$-1{\cdot}552 - 2(0{\cdot}724) = -3$$

Example 2 : *Solve the system of equations $4x + 2y + z = 14$, $x + 5y - z$, $xy + 8z = 20$ using Gauss – Seidel iteration method.*

Solution :

The given system is

$$4x + 2y + z = 14 \quad ...(1)$$

$$x + 5y - z = 10 \quad ...(2)$$

$$x + y + 8z = 20 \quad ...(3)$$

Clearly the coefficient matrix $\begin{pmatrix} 4 & 2 & 1 \\ 1 & 5 & -1 \\ 1 & 1 & 8 \end{pmatrix}$ is diagonally dominant.

Hence we can apply Gauss – Seidel method without any difficulty.

From (1), (2) and (3) we get

$$x = \frac{1}{4}(14 - 2y - z) \quad ...(4)$$

$$y = \frac{1}{5}(10 - x + z) \quad ...(5)$$

$$z = \frac{1}{8}(20 - x - y) \quad ...(6)$$

First Iteration :

Putting $y = 0$, $z = 0$ in (4) we get, $x = \frac{14}{4} = 3.5$.

Putting $x = 3.5$, $z = 0$ in (5) we get

$$y = \frac{1}{5}[10 - 3{\cdot}5 + 0] = 1{\cdot}3$$

Putting $x = 3{\cdot}5$, $y = 1{\cdot}3$ in (6) we get

$$z = \frac{1}{8}[20 - 3{\cdot}5 - 1{\cdot}3] = \frac{15{\cdot}2}{8} = 1{\cdot}9$$

∴ In the first iteration we get

$$x = 3{\cdot}5 \;,\; y = 1{\cdot}3 \;,\; z = 1.9$$

Second Iteration

Putting $y = 1{\cdot}3$, $z = 1{\cdot}9$ in (4) we get

$$x = \frac{1}{4}[14 - 2(1{\cdot}3) - 1{\cdot}9] = 2{\cdot}375$$

∴ In the first iteration we get

$x = -3$, $y = 0{\cdot}84$

Second Iteration

Putting $y = 0{\cdot}84$ in (3) we get

$$x = -3 + 2\,(0{\cdot}84) = -1{\cdot}32$$

Putting $x = -1{\cdot}32$ in (4) we get

$$y = \frac{1}{25}[15 - 2\,(-1{\cdot}32)] = 0{\cdot}7056$$

∴ In the second iteration we get

$x = -1{\cdot}32$, $y = 0{\cdot}7056$

Third Iteration

Putting $y = 0{\cdot}7056$ in (3) we get

$$x = -3 + 2\,(0{\cdot}7056) = -1{\cdot}589$$

Putting $x = -1{\cdot}589$ in (4) we get

$$y = \frac{1}{25}[15 - 2\,(-1{\cdot}589)] = 0{\cdot}727$$

∴ In the third iteration we get

$x = -1{\cdot}589$, $y = 0{\cdot}727$

Fourth Iteration

Putting $y = 0{\cdot}727$ in (3) we get

$$x = -3 + 2\,(0{\cdot}727) = -1{\cdot}546$$

Putting $x = -1{\cdot}546$ in (4) we get

$$y = \frac{1}{25}[15 - 2\,(-1{\cdot}546)] = 0{\cdot}724$$

∴ In the fourth iteration we get

$x = -1{\cdot}546$, $y = 0{\cdot}724$

Fifth Iteration

Putting $y = 0{\cdot}724$ in (3) we get

$$x = -3 + 2\,(0{\cdot}724) = -1{\cdot}552$$

Putting $x = -1{\cdot}552$ in (4) we get

$$y = \frac{1}{25}[15 - 2\,(-1{\cdot}552)] = 0{\cdot}724$$

∴ In the fifth iteration we get

$x = -1{\cdot}552$, $y = 0{\cdot}724$

Sixth Iteration

Putting $y = 0{\cdot}724$ in (3) we get

$$x = -3 + 2\,(0{\cdot}724)$$
$$= -1{\cdot}552$$

Note 3 : For the Gauss - Seidal method to converge quickly, the coefficient matrix must be diagonally dominant. If it is not so, we have to rearrange the equations in such a way that the coefficient matrix is diagonally dominant, then only we can apply Gauss - Seidal method.

E.g. 1: The coefficient matrix $\begin{pmatrix} 8 & -3 & 2 \\ 6 & 3 & 12 \\ 4 & 11 & -1 \end{pmatrix}$ can be made diagonally dominant by interchanging second and third rows as $\begin{pmatrix} 8 & -3 & 2 \\ 4 & 11 & -1 \\ 6 & 3 & 12 \end{pmatrix}$.

E.g. 2: Consider the system of equations

$$x_1 + 7x_2 - x_3 = 3$$
$$5x_1 + x_2 + x_3 = 9$$
$$-3x_1 + 2x_2 + 7x_3 = 17$$

The coefficient matrix $\begin{pmatrix} 1 & 7 & -1 \\ 5 & 1 & 1 \\ -3 & 2 & 7 \end{pmatrix}$ is not diagonally dominant.

But the given system can be rearranged like

$$7x_2 + x_1 - x_3 = 3$$
$$x_2 + 5x_1 + x_3 = 9$$
$$2x_2 - 3x_1 + 7x_3 = 17$$

Now the coefficient matrix $\begin{pmatrix} 7 & 1 & -1 \\ 1 & 5 & 1 \\ 2 & -3 & 7 \end{pmatrix}$ is diagonally dominant. W... solved the solution appear in the order x_2, x_1 and x_3.

Example 1 : *Solve by Gauss – Seidel method*

$$x - 2y = -3$$
$$2x + 25y = 15$$

[A.U. May ...]

Solution :

The given system is

$$x - 2y = -3 \quad \text{...(1)}$$
$$2x + 25y = 15 \quad \text{...(2)}$$

From (1) and (2) we get,

$$x = -3 + 2y \quad \text{...(3)}$$
$$y = \frac{1}{25}[15 - 2x] \quad \text{...(4)}$$

First Iteration :

Putting $y = 0$ in (3) we get, $x = -3$

Putting $x = -3$ in (4) we get

$$y = \frac{1}{25}[15 - 2(-3)] = 0{\cdot}84$$

Such system is often amenable to an iterative process in which the system is first rewritten in the form

$$x_1 = \frac{1}{a_{11}}(C_1 - a_{12}x_2 - a_{13}x_3 - \ldots\ldots - a_{1n}x_n) \qquad \ldots (1)$$

$$x_2 = \frac{1}{a_{22}}(C_1 - a_{21}x_1 - a_{23}x_3 - \ldots\ldots - a_{2n}x_n) \qquad \ldots (2)$$

$$x_3 = \frac{1}{a_{33}}(C_3 - a_{31}x_1 - a_{32}x_2 - \ldots\ldots - a_{3n}x_n) \qquad \ldots (3)$$

....

$$x_n = \frac{1}{a_{nn}}(C_n - a_{n1}x_1 - a_{n2}x_2 - \ldots\ldots - a_{n,\,n-1}x_{n-1}) \qquad \ldots (4)$$

First let us assume that $x_2 = x_3 = \ldots\ldots = x_n = 0$ in (1) and find x_1. Let it be x_1^*. Putting x_1^* for x_1 and $x_3 = x_4 = \ldots = x_n = 0$ in (2) we get the value for x_2 and let it be x_2^*. Putting x_1^* for x_1 and x_2^* for x_2 and $x_3 = x_4 = \ldots = x_n = 0$ in (3) we get the value for x_3 and let it be x_3^*. In this way we can find the first approximate values for $x_1, x_2, \ldots\ldots, x_n$. Similarly we can find the better approximate value of $x_1, x_2, \ldots\ldots, x_n$ by using the relation

$$x_1^* = \frac{1}{a_{11}}(C_1 - a_{12}x_2 - a_{13}x_3 \ldots\ldots - a_{1n}x_n)$$

$$x_2^* = \frac{1}{a_{22}}(C_2 - a_{21}x_1^* - a_{23}x_3 - \ldots\ldots - a_{2n}x_n)$$

$$x_3^* = \frac{1}{a_{33}}(C_3 - a_{31}x_1^* - a_{32}x_2^* - \ldots\ldots - a_{3n}x_n)$$

$$x_n^* = \frac{1}{a_{nn}}(C_n - a_{n1}x_1^* - a_{n2}x_2^* - a_{n3}x_3^* - \ldots\ldots - a_{n,\,n-1}x_{n-1}^*)$$

Note 1 : This method is very useful with less work for the given systems of equation whose augmented matrix have a large number of zero elements.

Note 2 : We say a matrix is diagonally dominant if the numerical value of the leading diagonal element in each row, is greater than or equal to the sum of the numerical values of the other elements in that row.

For example the matrix $\begin{pmatrix} 5 & 1 & -1 \\ 1 & 4 & 2 \\ 1 & -2 & 5 \end{pmatrix}$ *is diagonally dominant.*

But the matrix $\begin{pmatrix} 5 & 1 & -1 \\ 5 & 2 & 3 \\ 1 & -2 & 5 \end{pmatrix}$ *is not, since in the second row, the leading diagonal element 2 is less than the sum 8 of the other two elements viz., 5 and 3 in that row.*

Third approximation

$$\left.\begin{aligned} x_3 &= \frac{1}{20}(17 - y_2 + 2z_2) \\ y_3 &= \frac{1}{20}(-18 - 3x_2 + z_2) \\ z_3 &= \frac{1}{20}(25 - 2x_2 + 3y_2) \end{aligned}\right\}$$

Putting $x_2 = 1.02$, $y_2 = -0.965$, $z_2 = 1.03$, we get,

$$x_3 = \frac{1}{20}[17 + 0.965 + 2(1.03) \approx 1.00$$

$$y_3 = \frac{1}{20}[-18 - 3(1.02) + 1.03] \approx -1.0$$

$$z_3 = \frac{1}{20}[25 - 2(1.02) + 3(-0.965)] \approx 1.003$$

Fourth approximation

$$x_4 = \frac{1}{20}(17 - y_3 + 2z_3)$$

$$y_4 = \frac{1}{20}(-18 - 3x_3 + z_3)$$

$$z_4 = \frac{1}{20}(25 - 2x_3 + 3y_3)$$

Putting $x_3 = 1.00125$, $y_3 = -1.0015$, $z_3 = 1.0032$, we get

$$x_4 = \frac{1}{20}[17 + 1.0015 + 2(1.0032)] = 1.00039$$

$$y_4 = \frac{1}{20}[-18 - 3(1.00125) + 1.0032] = -1.00$$

$$z_4 = \frac{1}{20}[25 - 2(1.00125) + 3(-1.0015)]$$

$$= 0.99965$$

From 3rd and 4th approximation, we get

$$x = 1, \; y \approx -1, \; z \approx 1.$$

❑ GAUSS - SEIDEL ITERATIVE METHOD

Let the given system of equations be

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots + a_{1n}x_n &= C_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots + a_{2n}x_n &= C_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots\ldots + a_{3n}x_n &= C_3 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

$$x = \frac{8}{7} - \frac{1}{7}(1.50697) + \frac{1}{28}(1.8485)$$

$$x = 0.9936$$

$$\therefore x = 0.9936;\ y = 1.5070,\ z = 1.8485$$

Checking :

$$28x + 4y - z = 32$$

$$28(0.9936) + 4(1.5070) - 1.8425 = 32.0063$$

Example 3 : *Solve the system of equations by Gauss – elimination meth*

$$10x - 2y + 3z = 23$$
$$2x + 10y - 5z = -33$$
$$3x - 4y + 10z = 41$$

Solution :

Step 1 : Write the given system of equations in augmented matrix f

$$\begin{pmatrix} 10 & -2 & 3 & 23 \\ 2 & 10 & -5 & -33 \\ 3 & -4 & 10 & 41 \end{pmatrix}$$

Step 2 : $R_1 \to R_1 \div 10$

$$\begin{pmatrix} 1 & \frac{-1}{5} & \frac{3}{10} & \frac{23}{10} \\ 2 & 10 & -5 & -33 \\ 3 & -4 & 10 & 41 \end{pmatrix}$$

Step 3 : $R_2 \to R_2 - 2R_1,\quad R_3 \to R_3 - 3R_1$

$$\begin{pmatrix} 1 & \frac{-1}{5} & \frac{3}{10} & \frac{23}{10} \\ 0 & \frac{52}{5} & \frac{-28}{5} & \frac{-188}{5} \\ 0 & \frac{-17}{5} & \frac{91}{10} & \frac{341}{10} \end{pmatrix}$$

Step 4 : $R_2 \to R_2 \div \frac{52}{5}$

$$\begin{pmatrix} 1 & \frac{-1}{5} & \frac{3}{10} & \frac{23}{10} \\ 0 & 1 & \frac{-7}{13} & \frac{-47}{13} \\ 0 & \frac{-17}{5} & \frac{91}{10} & \frac{341}{10} \end{pmatrix}$$

Step 5 : $R_3 \to R_3 + \left(\frac{17}{5}\right) R_2$

$$\begin{pmatrix} 1 & \frac{-1}{5} & \frac{3}{10} & \frac{23}{10} \\ 0 & 1 & \frac{-7}{13} & \frac{-47}{13} \\ 0 & 0 & \frac{189}{26} & \frac{567}{26} \end{pmatrix}$$

Step 6: Delete the first row and first column and perform steps 2 to 5 on the resulting matrix.

$$\begin{pmatrix} 1 & \frac{-2}{3} & 1 & 2 \\ 0 & \frac{-1}{3} & 0 & -1 \\ 0 & \frac{-11}{3} & 2 & 1 \end{pmatrix} \quad \leftarrow \textbf{New Pivot}$$

$$\begin{pmatrix} 1 & \frac{-2}{3} & 1 & 2 \\ 0 & \frac{-11}{3} & 2 & 1 \\ 0 & \frac{-1}{3} & 0 & -1 \end{pmatrix}$$

R_3 and R_2 are interchanged to move the pivot to the top of new submatrix

$$\begin{pmatrix} 1 & \frac{-2}{3} & 1 & 2 \\ 0 & 1 & \frac{-6}{11} & \frac{-3}{11} \\ 0 & \frac{-1}{3} & 0 & -1 \end{pmatrix}$$

R_3 is divided by $\frac{-11}{3}$ to make the pivot as 1

$$\begin{pmatrix} 1 & \frac{-2}{3} & 1 & 2 \\ 0 & 1 & \frac{-6}{11} & \frac{-3}{11} \\ 0 & 0 & \frac{-2}{11} & \frac{-12}{11} \end{pmatrix} \quad R_3 \to R_3 + \frac{1}{3} R_2$$

Step 7: Delete first two rows and first two columns. Perform step 6 on the resulting matrix.

$$\begin{pmatrix} 1 & \frac{-2}{3} & 1 & 2 \\ 0 & 1 & \frac{-6}{11} & \frac{-3}{11} \\ 0 & 0 & \frac{-2}{11} & \frac{-12}{11} \end{pmatrix} \quad \leftarrow \text{New Pivot}$$

$$\begin{pmatrix} 1 & \frac{-2}{3} & 1 & 2 \\ 0 & 1 & \frac{-6}{11} & \frac{-3}{11} \\ 0 & 0 & 1 & 6 \end{pmatrix}$$

R_3 is divided by the new pivot.

Step 8: Use back substitution to find the solution to the system.

Here

$$x_3 = 6$$

$$x_2 - \frac{6}{11} x_3 = \frac{-3}{11}$$

$$x_2 = \frac{-3}{11} + \frac{6}{11}(6) = 3$$

$$x_1 - \frac{2}{3} x_2 + x_3 = 2$$

$$x_1 = 2 + \frac{2}{3} x_2 - x_3 = 2 + \frac{2}{3}(3) - 6 = -2$$

❑ Gauss Elimination Method

Basically the most effective direct solution techniques, currently used are applications of Gauss elimination, method which proposed over a century ago. In this method, the given system transformed into an equivalent system with upper-triangular coefficient matrix *i.e.*, a matrix in which all elements below the diagonal elements are zero which can be solved by back substitution. This method is clear from the following example.

Example 1 : ***Solve the following system by Gaussian elimination method***

$$\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ -3x_1 + 2x_2 - 3x_3 &= -6 \\ 2x_1 - 5x_2 + 4x_3 &= 5 \end{aligned}$$

Solution :

Step 1 : Write the given system in augmented matrix form.

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ -3 & 2 & -3 & -6 \\ 2 & -5 & 4 & 5 \end{array}\right)$$

Step 2 : From the first column with non-zero components (called the pivot column), select the component with the largest **absolute** value. This component is called the **pivot.**

$$\text{Pivot} \rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ -3 & 2 & -3 & -6 \\ 2 & -5 & 4 & 5 \end{array}\right)$$

Step 3 : Rearrange the rows to move the pivot element to the top of the column. Here we interchange the first and second row.

$$\text{Pivot} \rightarrow \left(\begin{array}{ccc|c} -3 & 2 & -3 & -6 \\ 1 & -1 & 1 & 1 \\ 2 & -5 & 4 & 5 \end{array}\right) \quad R_1 \sim R_2$$

Step 4 : Make the pivot as 1, by dividing the first row by the pivot.

$$\left(\begin{array}{ccc|c} 1 & \frac{-2}{3} & 1 & 2 \\ 1 & -1 & 1 & 1 \\ 2 & -5 & 4 & 5 \end{array}\right) \quad \frac{R_1}{3}$$

Step 5 : Add multiples of the first row to the other rows to make all other components in the pivot column equal to zero.

$$\left(\begin{array}{ccc|c} 1 & \frac{-2}{3} & 1 & 2 \\ 0 & \frac{-1}{3} & 0 & -1 \\ 0 & \frac{-11}{3} & 2 & 1 \end{array}\right) \quad \begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - 2R_1 \end{array}$$

until we get the root to the desired accuracy.

The order of convergence of Regula Falsi method is 1·618. (This may be assumed.)

Example 1. *Solve for a positive root of $x^3-4x+1=0$ by Regula Falsi method.*

Solution. Let $f(x)=x^3-4x+1=0$

$$f(1)=-2=-ve\,;\; f(2)=1=+ve\,,\; f(0)=1=+ve$$

$\therefore$ a root lies between 0 and 1

Another root lies between 1 and 2

We shall find the root that lies between 0 and 1

Here $a=0,\ b=1$

$$x_1=\frac{af(b)-bf(a)}{f(b)-f(a)}=\frac{0\times f(1)-1\times f(0)}{f(1)-f(0)}=\frac{-1}{-2-1}=0\cdot333333$$

$$f(x_1)=f\left(\tfrac{1}{3}\right)=\frac{1}{27}-\frac{4}{3}+1=-0\cdot2963$$

Now $f(0)$ and $f\left(\frac{1}{3}\right)$ are opposite in sign.

Hence the root lies between 0 and 1/3.

Hence $$x_2=\frac{0.f\left(\frac{1}{3}\right)-\frac{1}{3}f(0)}{f\left(\frac{1}{3}\right)-f(0)}$$

$$x_2=\frac{-\frac{1}{3}}{-1\cdot2963}=0\cdot25714$$

Now $f(x_2)=f(0\cdot25714)=-0\cdot011558=-ve$

$\therefore$ The root lies between 0 and 0·25714

$$x_3=\frac{0\times f(0\cdot25714)-0\cdot25714\,f(0)}{f(0\cdot25714)-f(0)}$$

$$=\frac{-0\cdot25714}{-1\cdot011558}=0\cdot25420$$

$$f(x_3)=f(0\cdot25420)=-0\cdot0003742$$

$\therefore$ The root lies between 0 and 0·25420

$$\therefore\quad x_4=\frac{0\times f(0\cdot25420)-0\cdot25420\times f(0)}{f(0\cdot25420)-f(0)}$$

$$=\frac{-0\cdot25420}{-1\cdot0003742}=0\cdot25410$$

$$f(x_4)=f(0\cdot25410)=-0\cdot000012936$$

The root lies between 0 and 0·25410

$$x_5 = \frac{0 \times f(0.25410) - 0.25410 \times f(0)}{f(0.25410) - f(0)}$$

$$= \frac{-0.25410}{-1.000012936} = 0.25410$$

Hence the root is 0.25410.

Example 2. *Find an approximate root of x log*$_{10}$ ... *position method.*

Solution. Let $f(x) = x \log_{10} x - 1.2$

$$f(1) = -1.2 = -ve; f(2) = 2 \times 0.301...$$

$$f(3) = 3 \times 0.47712 - 1.2 = 0.231364$$

Hence a root lies between 2 and 3.

$$\therefore \quad x_1 = \frac{2f(3) - 3f(2)}{f(3) - f(2)} = \frac{2 \times 0.23136 - 3 \times (-0.5...}{0.23136 + 0.5979...}$$

$$f(x_1) = f(2.7210) = -0.017104$$

The root lies between x_1 and 3.

$$x_2 = \frac{x_1 \times f(3) - 3 \times f(x_1)}{f(3) - f(x_1)}$$

$$= \frac{2.721014 \times 0.231364 - 3 \times (-...}{0.23136 + 0.01710...}$$

$$= \frac{0.68084}{0.24846} = 2.740211$$

$$f(x_2) = f(2.7402) = 2.7402 \times \log(2.7402) - 1.2$$

$$= -0.00038905$$

$\therefore$ The root lies between 2.740211 and 3

$$\therefore \quad x_3 = \frac{2.7402 \times f(3) - 3 \times f(2.7402)}{f(3) - f(2.7402)}$$

$$= \frac{2.7402 \times 0.23136 + 3 \times 0.0003...}{0.23136 + 0.00038905}$$

$$= \frac{0.63514}{0.23175} = 2.740627$$

$$f(2.7406) = 0.00011998$$

$\therefore$ The root lies between 2.740211 and 2.740627

$$x_4 = \frac{2.7402 \times f(2.7406) - 2.7406 \times f...}{f(2.7406) - f(2.7402)}$$

$$= \frac{2.7402 \times 0.00011998 + 2.7406 \times ...}{0.00011998 + 0.0003...}$$

$$= \frac{0.0013950}{0.00050903} = 2.7405$$

x_2

Scanned by CamScanner

x-coordinate of the point of intersection of this chord with the ... an approximate value for the root of $f(x)=0$. Setting $y=0$ in ... equation, we get

$$\frac{-f(a)}{x-a}=\frac{f(a)-f(b)}{a-b}$$

$$x[f(a)-f(b)]-af(a)+af(b)=-af(a)+bf(a)$$

$$x[f(a)-f(b)]=bf(a)-af(b)$$

$$\therefore \quad x_1=\frac{af(b)-bf(a)}{f(b)-f(a)}$$

This value of x_1 gives an approximate value of the ... $f(x)=0$. $(a<x_1<b)$

Now $f(x_1)$ and $f(a)$ are of opposite signs or $f(x_1)$ and $f(b)$ ... opposite signs.

If $f(x_1).f(a)<0$, then x_2 lies between x_1 and a.

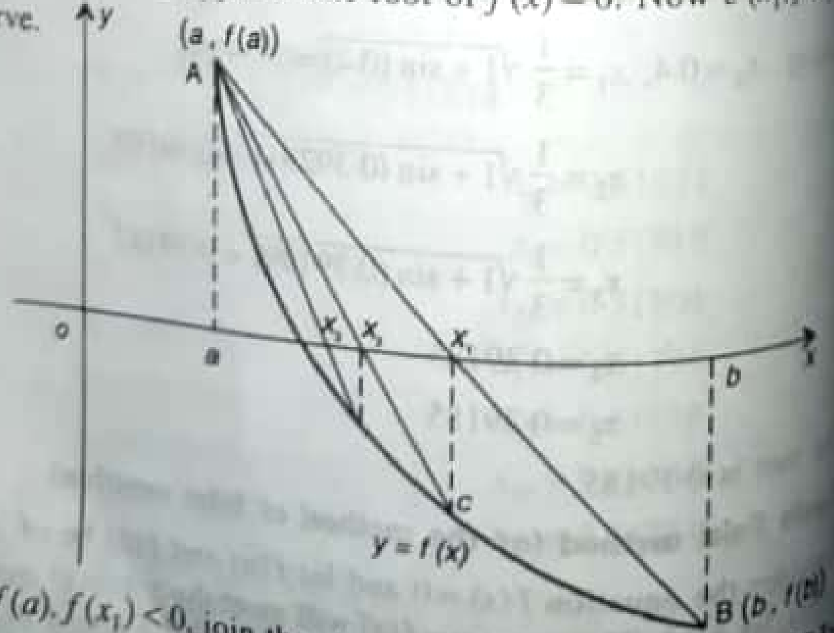
Hence $x_2=\dfrac{af(x_1)-x_1f(a)}{f(x_1)-f(a)}$

In the same way, we get $x_3, x_4, ...$

This sequence $x_1, x_2, x_3, ...$ will converge to the required ... practice, we get x_i and x_{i+1} such that $|x_i-x_{i+1}|<\varepsilon$, the ... accuracy.

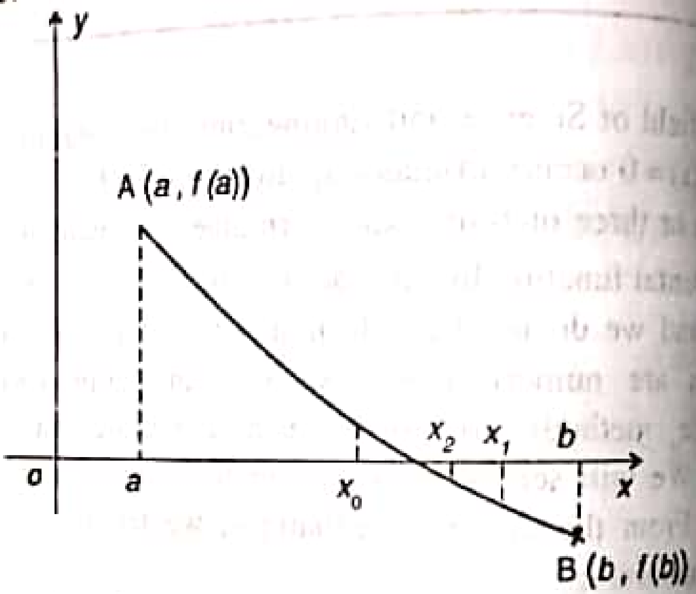
3.3.1. Geometrical interpretation

If $A(a, f(a))$ and $B(b, f(b))$ are two points on $y=f(x)$ su... $f(a)$ and $f(b)$ are opposite in sign, then the chord AB meets x-... $x=x_1$. This x_1 is the approximate root of $f(x)=0$. Now $c(x_1, f(x_1))$ ... the curve.



If $f(a).f(x_1)<0$, join the chord AC which cuts x-axis at $x=x_2$ ... x_2 is the second approximate root of $f(x)=0$. This process is co...

b and take the root as $x_1 = \frac{x_0 + b}{2}$. Now $f(x_1)$ is negative ... 3·1). Hence the root lies between x_0 and x_1 and let the (approximate) $x_2 = \frac{x_0 + x_1}{2}$. Now $f(x_2)$ is negative as in the Fig. ... the root lies between x_0 and x_2 and let $x_3 = \frac{x_0 + x_2}{2}$ and so on. In ... taking the mid-point of the range as the approximate root, we ... sequence of approximate roots $x_0, x_1, x_2, \ldots$ whose limit of converg... the exact root. However, depending on the precision required, we ... process after some steps. Though simple, the convergence of this ... is *slow but sure*.



Note. After n bisections, the length of the subinterval which contain... $\frac{b-a}{2^n}$. If the error is to be made less than a small quantity ε ...

$$\frac{b-a}{2^n} < \varepsilon. \text{ That is, } 2^n > \frac{b-a}{\varepsilon}$$

The number of iterations n should be greater than $\frac{\log\left(\frac{b-a}{\varepsilon}\right)}{\log 2}$

Example 1. *Find the positive root of $x^3 - x = 1$ correct to ... decimal places by bisection method.*

Solution. Let $f(x) = x^3 - x - 1$

Here, $f(0) = -1 = -ve$ and $f(1) = -ve$

$f(2) = 5 = +ve$. Hence a root lies between 1 and 2. We can take the ... as (1, 2) and proceed. We can still shorten the range.

and $f(1{\cdot}5) = 0{\cdot}8750 = +ve$

$f(1$ ...

3

The Solution of Numerical Algebraic and Transcendental Equations

3.1. In the field of Science and Engineering, the solution of equations of the form $f(x)=0$ occurs in many applications. If $f(x)$ is a polynomial of degree two or three or four, exact formulae are available. But, if $f(x)$ is a transcendental function like $a+be^x+c\sin x+d\log x$ etc., the solution is not exact and we do not have formulae to get the solutions. When the coefficients are numerical values, we can adopt various numerical approximate methods to solve such algebraic and transcendental equations. We will see below some methods of solving such numerical equations. From the theory of equations, we recall to our memory the following theorem:

If $f(x)$ is continuous in the interval (a, b) and if $f(a)$ and $f(b)$ are of opposite signs, then the equation $f(x)=0$ will have atleast one real root between a and b.

3.1.1. The Bisection method (or BOLZANO's method) (or Interval halving method)

AIM: Suppose we have an equation of the form $f(x)=0$ whose solution in the range (a, b) is to be searched. We also assume that $f(x)$ is continuous and it can be algebraic or transcendental. If $f(a)$ and $f(b)$ are of opposite signs, atleast one real root between a and b should exist. For convenience, let $f(a)$ be positive and $f(b)$ be negative. Then atleast one root exists between a and b. As a first approximation, we assume that root to be $x_0=\dfrac{a+b}{2}$ (mid point of the ends of the range). Now, find the sign of $f(x_0)$. If $f(x_0)$ is negative, the root lies between a and x_0. If $f(x_0)$ is positive, the root lies between x_0 and b. Any one of this is true. Suppose $f(x_0)$ is positive as shown in the Fig. 3·1, then the root lies between x_0 and

Value of y at $x = 500$ is 115.

Note. The equation is not unique since the answer depends upon our approach. One can give a number approximating straight lines. Hence, graphical approach is not accurate. Hence we will go to analytical approach rather than graphical approach.

EXERCISE 1.1

1. Find the relation of the form $y = \frac{a}{x} + b$ from the data using the graph:

x :	1·95	2·46	3	3·44	3·96
y :	67	63	59·6	58	56

2. Find the relation of the form $y = a + \frac{b}{x}$ given

x :	36·8	31·5	26·3	21·0	15·8	12·6	8·4
y :	12·5	12·9	13·1	13·3	14·1	14·5	16·3

3. Plot the points and obtain a st. line fit from the graph.

x :	1	2	3	4	5	6
y :	1200	900	600	200	110	50

4. For the following data, fit a curve of the form, $yx^n = k$ from the data given below:

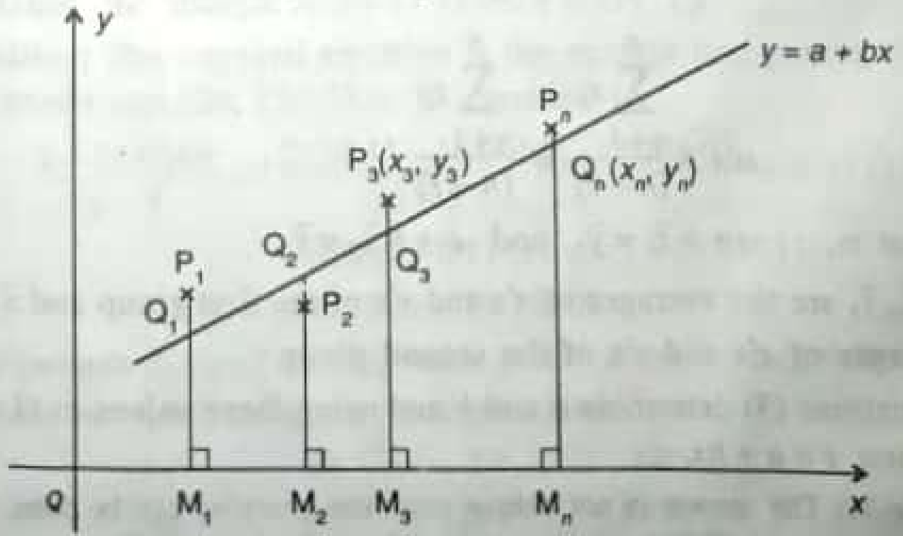
y :	15	30	60	100	200	400
x :	10	5·6	3·2	2·1	1·2	0·7

1.4. Evaluation of constants by the method of group averages (To fit a straight line)

Let (x_i, y_i), $i = 1, 2, \ldots, n$ be n sets of observations and the law relating x and y be

$$y = a + bx \qquad \ldots(1)$$

We assume that the law is a linear one (we do not prove it is linear). We want to determine a and b. By assumption $y = a + bx$ is the law, it does not mean the points lie on the st. line. It is only an approximating st. line and it is possible that no point may lie on the line.



Taking logarithm on both sides,

$\log_{10} y = \log_{10} a + n \log_{10} x$

i.e., $Y = A + nX$ where $Y = \log_{10} y$, $A = \log_{10} a$ and $X = \log_{10} x$

Now A and n can be found as explained earlier.

Hence, a and n are known.

(2) Law of the pattern $y = a e^{bx}$ where a, b are constants.

Taking logarithm on both sides,

$\log_{10} y = \log_{10} a + bx$

i.e., $Y = A + bx$ where $\log_{10} y = Y$, $A = \log_{10} a$

After getting A, b we get again a and b.

Hence, $y = a e^{bx}$ is known.

(3) Law of the pattern $y = a + bx^n$ where n is known and a, b are unknown constants.

Setting $x^n = u$, we have $y = a + bu$, which is linear in y and u. Since x is given, u can be found out. The linear fit $y = a + bu$ can be got as explained earlier.

The forms $y = a + bx^2$, $y = a + b\sqrt{x}$, $y = a + \dfrac{b}{x}$ are all of this type explained above.

In general, getting an emperical equation, depends upon :

(*i*) the determination of the form of the equation, and

(*ii*) the evaluation of the constants in the equation.

Normally, the form of the equation is agreed upon from the data and then the constants are evaluated.

In this chapter, given the form of the equation, we will try to evaluate the constants.

Example 1. *Plot the points on the graph and obtain a relation of the form $y = ax + b$ given the following data. Find y at $x = 500$.*

x	225	300	430	560	600
y	60	75	100	125	145

Solution. Since the form is already accepted to be linear, we plot the points on the graph-sheet and draw a smooth straight line which is only approximate.

$$\text{Slope} = \frac{75 - 60}{300 - 225} = 0{\cdot}2$$

Hence $y = 0{\cdot}2x + b$

Substituting, $60 = 0{\cdot}2(225) + b \quad \therefore \quad b = 15$

Hence the rough form is $y = 0{\cdot}2x + 15$